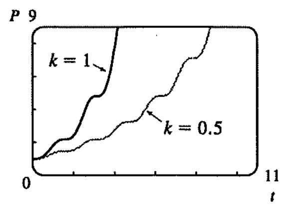
HW #10, Sec. 9.5 Solutions

$\ln P = \frac{k}{2}t + \frac{k}{4r}\sin(2(rt-\phi)) + \ln P_0 + \frac{k}{4r}\sin 2\phi$. Simplifying, we get

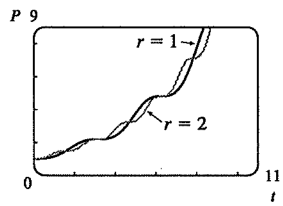
$\ln \frac{P}{P_0} = \frac{k}{2}t + \frac{k}{4r}[\sin(2(rt-\phi)) + \sin 2\phi] = f(t)$, or $P(t) = P_0 e^{f(t)}$.

(b) An increase in k stretches the graph of P vertically while maintaining $P(0) = P_0$.



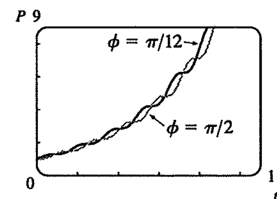
Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$

An increase in r compresses the graph of P horizontally—similar to changing the period in Exercise 23.



Comparing values of r with $P_0 = 1$, $k = 0.5$, and $\phi = \pi/2$

As in Exercise 23, a change in ϕ only makes slight adjustments in the growth of P, as shown in the figure.



Comparing values of ϕ with $P_0 = 1$, $k = 0.5$, and $r = 2$

$f'(t) = k/2 + [k/(4r)][2r\cos(2(rt-\phi))] = (k/2)[1+\cos(2(rt-\phi))] \geq 0$. Since $P(t) = P_0 e^{f(t)}$, we have $P'(t) = P_0 f'(t) e^{f(t)} \geq 0$, with equality only when $\cos(2(rt-\phi)) = -1$; that is, when $rt - \phi$ is an odd multiple of $\frac{\pi}{2}$. Therefore, $P(t)$ is an increasing function on $(0,\infty)$. P can also be written as $P(t) = P_0 e^{kt/2} e^{(k/4r)[\sin(2(rt-\phi))+\sin 2\phi]}$. The second exponential oscillates between $e^{(k/4r)(1+\sin 2\phi)}$ and $e^{(k/4r)(-1+\sin 2\phi)}$, while the first one, $e^{kt/2}$, grows without bound. So $\lim\limits_{t\to\infty} P(t) = \infty$.

25. By Equation 7, $P(t) = \frac{K}{1+Ae^{-kt}}$. By comparison, if $c = (\ln A)/k$ and $u = \frac{1}{2}k(t-c)$, then

$$1 + \tanh u = 1 + \frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{e^u + e^{-u}}{e^u + e^{-u}} + \frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{2e^u}{e^u + e^{-u}} \cdot \frac{e^{-u}}{e^{-u}} = \frac{2}{1+e^{-2u}}$$

and $e^{-2u} = e^{-k(t-c)} = e^{kc}e^{-kt} = e^{\ln A}e^{-kt} = Ae^{-kt}$, so

$$\tfrac{1}{2}K\left[1 + \tanh\left(\tfrac{1}{2}k(t-c)\right)\right] = \frac{K}{2}[1+\tanh u] = \frac{K}{2}\cdot\frac{2}{1+e^{-2u}} = \frac{K}{1+e^{-2u}} = \frac{K}{1+Ae^{-kt}} = P(t).$$

9.5 Linear Equations

1. $y' + x\sqrt{y} = x^2$ is not linear since it cannot be put into the standard form (1), $y' + P(x)\,y = Q(x)$.

2. $y' - x = y\tan x \iff y' + (-\tan x)y = x$ is linear since it can be put into the standard form (1), $y' + P(x)\,y = Q(x)$.

3. $ue^t = t + \sqrt{t}\,\frac{du}{dt} \iff \sqrt{t}\,u' - e^t u = -t \iff u' - \frac{e^t}{\sqrt{t}}u = -\sqrt{t}$ is linear since it can be put into the standard form, $u' + P(t)\,u = Q(t)$.

4. $\frac{dR}{dt} + t\cos R = e^{-t} \iff R' + t\cos R = e^{-t}$ is not linear since it cannot be put into the standard form $R' + P(t)\,R = Q(t)$.

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5. Comparing the given equation, $y' + y = 1$, with the general form, $y' + P(x)\,y = Q(x)$, we see that $P(x) = 1$ and the integrating factor is $I(x) = e^{\int P(x)\,dx} = e^{\int 1\,dx} = e^x$. Multiplying the differential equation by $I(x)$ gives

$$e^x y' + e^x y = e^x \;\Rightarrow\; (e^x y)' = e^x \;\Rightarrow\; e^x y = \int e^x\,dx \;\Rightarrow\; e^x y = e^x + C \;\Rightarrow\; \frac{e^x y}{e^x} = \frac{e^x}{e^x} + \frac{C}{e^x} \;\Rightarrow$$

$y = 1 + Ce^{-x}$.

6. $y' - y = e^x \;\Leftrightarrow\; y' + (-1)y = e^x \;\Rightarrow\; P(x) = -1.$ $I(x) = e^{\int P(x)\,dx} = e^{\int -1\,dx} = e^{-x}$. Multiplying the original differential equation by $I(x)$ gives $e^{-x}y' - e^{-x}y = e^0 \;\Rightarrow\; (e^{-x}y)' = 1 \;\Rightarrow\; e^{-x}y = \int 1\,dx \;\Rightarrow$

$$e^{-x}y = x + C \;\Rightarrow\; y = \frac{x + C}{e^{-x}} \;\Rightarrow\; y = xe^x + Ce^x.$$

7. $y' = x - y \;\Rightarrow\; y' + y = x\ (\star)$. $I(x) = e^{\int P(x)\,dx} = e^{\int 1\,dx} = e^x$. Multiplying the differential equation $(\star)$ by $I(x)$ gives $e^x y' + e^x y = xe^x \;\Rightarrow\; (e^x y)' = xe^x \;\Rightarrow\; e^x y = \int xe^x\,dx \;\Rightarrow\; e^x y = xe^x - e^x + C$ [by parts] $\Rightarrow$ $y = x - 1 + Ce^{-x}$ [divide by e^x].

8. $4x^3 y + x^4 y' = \sin^3 x \;\Rightarrow\; (x^4 y)' = \sin^3 x \;\Rightarrow\; x^4 y = \int \sin^3 x\,dx \;\Rightarrow$

$$x^4 y = \int \sin x\,(1 - \cos^2 x)\,dx = \int (1 - u^2)(-du) \quad \begin{bmatrix} u = \cos x, \\ du = -\sin x\,dx \end{bmatrix}$$

$$= \int (u^2 - 1)\,du = \tfrac{1}{3}u^3 - u + C = \tfrac{1}{3}u(u^2 - 3) + C = \tfrac{1}{3}\cos x\,(\cos^2 x - 3) + C \;\Rightarrow$$

$$y = \frac{1}{3x^4}\cos x\,(\cos^2 x - 3) + \frac{C}{x^4}$$

9. Since $P(x)$ is the derivative of the coefficient of y' [$P(x) = 1$ and the coefficient is x], we can write the differential equation $xy' + y = \sqrt{x}$ in the easily integrable form $(xy)' = \sqrt{x} \;\Rightarrow\; xy = \tfrac{2}{3}x^{3/2} + C \;\Rightarrow\; y = \tfrac{2}{3}\sqrt{x} + C/x$.

10. $2xy' + y = 2\sqrt{x} \;\Rightarrow\; y' + \dfrac{1}{2x}y = \dfrac{1}{\sqrt{x}}$ $[x > 0]$ $\Rightarrow\; P(x) = \dfrac{1}{2x}$.

$I(x) = e^{\int P(x)\,dx} = e^{\int 1/(2x)\,dx} = e^{(1/2)\ln|x|} = (e^{\ln x})^{1/2} = \sqrt{x}$. Multiplying the differential equation by $I(x)$ gives

$$\sqrt{x}\,y' + \frac{1}{2\sqrt{x}}\,y = 1 \;\Rightarrow\; (\sqrt{x}\,y)' = 1 \;\Rightarrow\; \sqrt{x}\,y = \int 1\,dx \;\Rightarrow\; \sqrt{x}\,y = x + C \;\Rightarrow\; y = \frac{x + C}{\sqrt{x}}.$$

11. $xy' - 2y = x^2 \;\Rightarrow\; y' - \dfrac{2}{x}y = x \;\Rightarrow\; P(x) = -\dfrac{2}{x}$.

$I(x) = e^{\int P(x)\,dx} = e^{\int -2/x\,dx} = e^{-2\ln x}$ $[x > 0]$ $= x^{-2} = \dfrac{1}{x^2}$. Multiplying the differential equation by $I(x)$ gives

$$\frac{1}{x^2}y' - \frac{2}{x^3}y = \frac{1}{x} \;\Rightarrow\; \left(\frac{1}{x^2}y\right)' = \frac{1}{x} \;\Rightarrow\; \frac{1}{x^2}y = \int \frac{1}{x}\,dx \;\Rightarrow\; \frac{1}{x^2}y = \ln x + C \;\Rightarrow\; y = x^2(\ln x + C).$$

12. $y' - 3x^2 y = x^2 \;\Rightarrow\; P(x) = -3x^2$. $I(x) = e^{\int P(x)\,dx} = e^{\int -3x^2\,dx} = e^{-x^3}$. Multiplying the differential equation by $I(x)$ gives $e^{-x^3}y' - 3x^2 e^{-x^3}y = x^2 e^{-x^3} \;\Rightarrow\; \left(e^{-x^3}y\right)' = x^2 e^{-x^3} \;\Rightarrow\; e^{-x^3 y} = \int x^2 e^{-x^3}\,dx \;\Rightarrow$

$$e^{-x^3}y = -\tfrac{1}{3}e^{-x^3} + C \quad \begin{bmatrix} \text{by substitution} \\ \text{with } u = -x^3 \end{bmatrix} \;\Rightarrow\; y = -\tfrac{1}{3} + Ce^{x^3}.$$

13. $t^2\,\dfrac{dy}{dt} + 3ty = \sqrt{1+t^2} \;\Rightarrow\; y' + \dfrac{3}{t}\,y = \dfrac{\sqrt{1+t^2}}{t^2} \;\Rightarrow\; P(t) = \dfrac{3}{t}$.

$I(t) = e^{\int P(t)\,dt} = e^{\int 3/t\,dt} = e^{3\ln t}$ $[t>0]$ $= t^3$. Multiplying by t^3 gives $t^3y' + 3t^2y = t\sqrt{1+t^2} \;\Rightarrow$ $(t^3y)' = t\sqrt{1+t^2} \;\Rightarrow\; t^3y = \int t\sqrt{1+t^2}\,dt \;\Rightarrow\; t^3y = \frac{1}{3}(1+t^2)^{3/2} + C \;\Rightarrow\; y = \frac{1}{3}t^{-3}(1+t^2)^{3/2} + Ct^{-3}$.

14. $t\ln t\,\dfrac{dr}{dt} + r = te^t \;\Rightarrow\; \dfrac{dr}{dt} + \dfrac{1}{t\ln t}\,r = \dfrac{e^t}{\ln t}$. $I(t) = e^{\int dt/(t\ln t)} = e^{\ln(\ln t)} = \ln t$. Multiplying by $\ln t$ gives

$\ln t\,\dfrac{dr}{dt} + \dfrac{1}{t}\,r = e^t \;\Rightarrow\; [(\ln t)r]' = e^t \;\Rightarrow\; (\ln t)r = e^t + C \;\Rightarrow\; r = \dfrac{e^t + C}{\ln t}$.

15. $y' + y\cos x = x \;\Rightarrow\; P(x) = \cos x$. $I(x) = e^{\int P(x)\,dx} = e^{\int \cos x\,dx} = e^{\sin x}$. Multiplying the differential equation by $I(x)$ gives $e^{\sin x}y' + e^{\sin x}\cos x\cdot y = xe^{\sin x} \;\Rightarrow\; (e^{\sin x}y)' = xe^{\sin x} \;\Rightarrow\; e^{\sin x}y = \int xe^{\sin x}\,dx + C \;\Rightarrow$ $y = e^{-\sin x}\int xe^{\sin x}\,dx + Ce^{-\sin x}$. *Note:* $f(x) = xe^{\sin x}$ has an antiderivative F that is *not* an elementary function [see Section 7.5].

16. $y' + 2xy = x^3\,e^{x^2} \;\Rightarrow\; P(x) = 2x$. $I(x) = e^{\int P(x)\,dx} = e^{\int 2x\,dx} = e^{x^2}$. Multiplying the differential equation by $I(x)$ gives $e^{x^2}y' + 2xe^{x^2}y = x^3e^{2x^2} \;\Rightarrow\; \left(e^{x^2}y\right)' = x^3e^{2x^2} \;\Rightarrow\; e^{x^2}y = \int x^3e^{2x^2}\,dx \;\Rightarrow$

$$e^{x^2}y = \tfrac{1}{4}x^2e^{2x^2} - \int \tfrac{1}{2}xe^{2x^2}\,dx \quad \begin{bmatrix} u = \frac{1}{4}x^2, & dv = 4xe^{2x^2}\,dx \\ du = \frac{1}{2}x\,dx, & v = e^{2x^2} \end{bmatrix} \;\Rightarrow$$

$e^{x^2}y = \frac{1}{4}x^2e^{2x^2} - \frac{1}{2}\int e^z\left(\frac{1}{4}\,dz\right)$ $[z = 2x^2, dz = 4x\,dx]$ $\Rightarrow\; e^{x^2}y = \frac{1}{4}x^2e^{2x^2} - \frac{1}{8}e^{2x^2} + C \;\Rightarrow$

$y = \frac{1}{4}x^2e^{x^2} - \frac{1}{8}e^{x^2} + Ce^{-x^2}$.

17. $xy' + y = 3x^2 \;\Rightarrow\; (xy)' = 3x^2 \;\Rightarrow\; xy = \int 3x^2dx \;\Rightarrow\; xy = x^3 + C \;\Rightarrow\; y = x^2 + \dfrac{C}{x}$. Since $y(1) = 4$,

$4 = 1^2 + \dfrac{C}{1} \;\Rightarrow\; C = 3$, so $y = x^2 + \dfrac{3}{x}$.

18. $xy' - 2y = 2x \;\Rightarrow\; y' - \dfrac{2}{x}y = 2$ $(\star)$. $I(x) = e^{-2\int 1/x\,dx} = e^{-2\ln|x|} = e^{\ln|x|^{-2}} = |x|^{-2} = x^{-2}$. Multiplying $(\star)$ by $I(x)$ gives $x^{-2}y' - \dfrac{2x^{-2}}{x}y = 2x^{-2} \;\Rightarrow\; (x^{-2}y)' = 2x^{-2} \;\Rightarrow\; x^{-2}y = \int 2x^{-2}\,dx \;\Rightarrow\; x^{-2}y = -2x^{-1} + C \;\Rightarrow$ $y = -2x + Cx^2$. Since $y(2) = 0$, $0 = -2(2) + C(2)^2 \;\Rightarrow\; C = 1$, so $y = x^2 - 2x$.

19. $x^2y' + 2xy = \ln x \;\Rightarrow\; (x^2y)' = \ln x \;\Rightarrow\; x^2y = \int \ln x\,dx \;\Rightarrow\; x^2y = x\ln x - x + C$ [by parts]. Since $y(1) = 2$, $1^2(2) = 1\ln 1 - 1 + C \;\Rightarrow\; 2 = -1 + C \;\Rightarrow\; C = 3$, so $x^2y = x\ln x - x + 3$, or $y = \dfrac{1}{x}\ln x - \dfrac{1}{x} + \dfrac{3}{x^2}$.

20. $t^3\,\dfrac{dy}{dt} + 3t^2y = \cos t \;\Rightarrow\; (t^3y)' = \cos t \;\Rightarrow\; t^3y = \int \cos t\,dt \;\Rightarrow\; t^3y = \sin t + C$. Since $y(\pi) = 0$,

$\pi^3(0) = \sin\pi + C \;\Rightarrow\; C = 0$, so $t^3y = \sin t$, or $y = \dfrac{\sin t}{t^3}$.

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21. $t\,\frac{du}{dt} = t^2 + 3u \;\Rightarrow\; u' - \frac{3}{t}u = t \;(\star)$. $I(t) = e^{\int -3/t\,dt} = e^{-3\ln|t|} = (e^{\ln|t|})^{-3} = t^{-3} \;\; [t > 0] = \frac{1}{t^3}$. Multiplying $(\star)$ by $I(t)$ gives $\frac{1}{t^3}\,u' - \frac{3}{t^4}u = \frac{1}{t^2} \;\Rightarrow\; \left(\frac{1}{t^3}u\right)' = \frac{1}{t^2} \;\Rightarrow\; \frac{1}{t^3}\,u = \int \frac{1}{t^2}\,dt \;\Rightarrow\; \frac{1}{t^3}\,u = -\frac{1}{t} + C$. Since $u(2) = 4$, $\frac{1}{2^3}(4) = -\frac{1}{2} + C \;\Rightarrow\; C = 1$, so $\frac{1}{t^3}\,u = -\frac{1}{t} + 1$, or $u = -t^2 + t^3$.

22. $xy' + y = x\ln x \;\Rightarrow\; (xy)' = x\ln x \;\Rightarrow\; xy = \int x\ln x\,dx \;\Rightarrow\; xy = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C \;\; \left[\begin{matrix}\text{by parts}\\ \text{with } u = \ln x\end{matrix}\right] \;\Rightarrow$ $y = \frac{1}{2}x\ln x - \frac{1}{4}x + \frac{C}{x}$. $y(1) = 0 \;\Rightarrow\; 0 = 0 - \frac{1}{4} + C \;\Rightarrow\; C = \frac{1}{4}$, so $y = \frac{1}{2}x\ln x - \frac{1}{4}x + \frac{1}{4x}$.

23. $xy' = y + x^2\sin x \;\Rightarrow\; y' - \frac{1}{x}\,y = x\sin x$. $I(x) = e^{\int(-1/x)\,dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$.

Multiplying by $\frac{1}{x}$ gives $\frac{1}{x}\,y' - \frac{1}{x^2}\,y = \sin x \;\Rightarrow\; \left(\frac{1}{x}\,y\right)' = \sin x \;\Rightarrow\; \frac{1}{x}\,y = -\cos x + C \;\Rightarrow\; y = -x\cos x + Cx$.

$y(\pi) = 0 \;\Rightarrow\; -\pi\cdot(-1) + C\pi = 0 \;\Rightarrow\; C = -1$, so $y = -x\cos x - x$.

24. $(x^2+1)\,\frac{dy}{dx} + 3x(y-1) = 0 \;\Rightarrow\; (x^2+1)\,y' + 3xy = 3x \;\Rightarrow\; y' + \frac{3x}{x^2+1}\,y = \frac{3x}{x^2+1}$.

$I(x) = e^{\int 3x/(x^2+1)\,dx} = e^{(3/2)\ln|x^2+1|} = \left(e^{\ln(x^2+1)}\right)^{3/2} = (x^2+1)^{3/2}$. Multiplying by $(x^2+1)^{3/2}$ gives

$(x^2+1)^{3/2}\,y' + 3x(x^2+1)^{1/2}\,y = 3x(x^2+1)^{1/2} \;\Rightarrow\; \left[(x^2+1)^{3/2}y\right]' = 3x(x^2+1)^{1/2} \;\Rightarrow$

$(x^2+1)^{3/2}\,y = \int 3x(x^2+1)^{1/2}\,dx = (x^2+1)^{3/2} + C \;\Rightarrow\; y = 1 + C(x^2+1)^{-3/2}$. Since $y(0) = 2$, we have $2 = 1 + C(1) \;\Rightarrow\; C = 1$ and hence, $y = 1 + (x^2+1)^{-3/2}$.

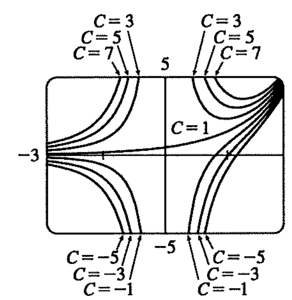
25. $xy' + 2y = e^x \;\Rightarrow\; y' + \frac{2}{x}\,y = \frac{e^x}{x}$.

$I(x) = e^{\int(2/x)\,dx} = e^{2\ln|x|} = \left(e^{\ln|x|}\right)^2 = |x|^2 = x^2$.

Multiplying by $I(x)$ gives $x^2\,y' + 2xy = xe^x \;\Rightarrow\; (x^2y)' = xe^x \;\Rightarrow$

$x^2y = \int xe^x\,dx = (x-1)e^x + C \;\;$ [by parts] $\;\Rightarrow$

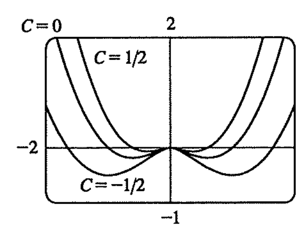
$y = [(x-1)e^x + C]/x^2$. The graphs for $C = -5, -3, -1, 1, 3, 5$, and 7 are shown. $C = 1$ is a transitional value. For $C < 1$, there is an inflection point and for $C > 1$, there is a local minimum. As $|C|$ gets larger, the "branches" get further from the origin.



26. $xy' = x^2 + 2y \;\Leftrightarrow\; xy' - 2y = x^2 \;\Leftrightarrow\; y' - \frac{2}{x}\,y = x$.

$I(x) = e^{\int -2/x\,dx} = e^{-2\ln|x|} = (e^{\ln|x|})^{-2} = |x|^{-2} = \frac{1}{x^2}$. Multiplying by $I(x)$ gives $\frac{1}{x^2}\,y' - \frac{2}{x^3}\,y = \frac{1}{x} \;\Rightarrow\; \left(\frac{1}{x^2}\,y\right)' = \frac{1}{x} \;\Rightarrow\; \frac{1}{x^2}\,y = \int \frac{1}{x}\,dx \;\Rightarrow$

$\frac{1}{x^2}\,y = \ln|x| + C \;\Rightarrow\; y = (\ln|x| + C)x^2$. For all values of C, as $|x| \to 0$,



37. $y(0) = 0$ kg. Salt is added at a rate of $\left(0.4\ \frac{\text{kg}}{\text{L}}\right)\left(5\ \frac{\text{L}}{\text{min}}\right) = 2\ \frac{\text{kg}}{\text{min}}$. Since solution is drained from the tank at a rate of 3 L/min, but salt solution is added at a rate of 5 L/min, the tank, which starts out with 100 L of water, contains $(100 + 2t)$ L of liquid after t min. Thus, the salt concentration at time t is $\frac{y(t)}{100+2t}\ \frac{\text{kg}}{\text{L}}$. Salt therefore leaves the tank at a rate of $\left(\frac{y(t)}{100+2t}\ \frac{\text{kg}}{\text{L}}\right)\left(3\ \frac{\text{L}}{\text{min}}\right) = \frac{3y}{100+2t}\ \frac{\text{kg}}{\text{min}}$. Combining the rates at which salt enters and leaves the tank, we get $\frac{dy}{dt} = 2 - \frac{3y}{100+2t}$. Rewriting this equation as $\frac{dy}{dt} + \left(\frac{3}{100+2t}\right) y = 2$, we see that it is linear.

$$I(t) = \exp\left(\int \frac{3\,dt}{100+2t}\right) = \exp\left(\tfrac{3}{2}\ln(100+2t)\right) = (100+2t)^{3/2}$$

Multiplying the differential equation by $I(t)$ gives $(100+2t)^{3/2}\frac{dy}{dt} + 3(100+2t)^{1/2}y = 2(100+2t)^{3/2} \Rightarrow$ $[(100+2t)^{3/2}y]' = 2(100+2t)^{3/2} \Rightarrow (100+2t)^{3/2}y = \frac{4}{5}(100+2t)^{5/2} + C \Rightarrow$ $y = \frac{2}{5}(100+2t) + C(100+2t)^{-3/2}$. Now $0 = y(0) = \frac{2}{5}(100) + C \cdot 100^{-3/2} = 40 + \frac{1}{1000}C \Rightarrow C = -40{,}000$, so $y = \left[\frac{2}{5}(100+2t) - 40{,}000(100+2t)^{-3/2}\right]$ kg. From this solution (no pun intended), we calculate the salt concentration at time t to be $C(t) = \frac{y(t)}{100+2t} = \left[\frac{-40{,}000}{(100+2t)^{5/2}} + \frac{2}{5}\right]\frac{\text{kg}}{\text{L}}$. In particular, $C(20) = \frac{-40{,}000}{140^{5/2}} + \frac{2}{5} \approx 0.2275\ \frac{\text{kg}}{\text{L}}$ and $y(20) = \frac{2}{5}(140) - 40{,}000(140)^{-3/2} \approx 31.85$ kg.

38. Let $y(t)$ denote the amount of chlorine in the tank at time t (in seconds). $y(0) = (0.05\ \text{g/L})\,(400\ \text{L}) = 20$ g. The amount of liquid in the tank at time t is $(400 - 6t)$ L since 4 L of water enters the tank each second and 10 L of liquid leaves the tank each second. Thus, the concentration of chlorine at time t is $\frac{y(t)}{400-6t}\ \frac{\text{g}}{\text{L}}$. Chlorine doesn't enter the tank, but it leaves at a rate of $\left[\frac{y(t)}{400-6t}\ \frac{\text{g}}{\text{L}}\right]\left[10\ \frac{\text{L}}{\text{s}}\right] = \frac{10\,y(t)}{400-6t}\ \frac{\text{g}}{\text{s}} = \frac{5\,y(t)}{200-3t}\ \frac{\text{g}}{\text{s}}$. Therefore, $\frac{dy}{dt} = -\frac{5y}{200-3t} \Rightarrow \int\frac{dy}{y} = \int\frac{-5\,dt}{200-3t} \Rightarrow$ $\ln y = \frac{5}{3}\ln(200-3t) + C \Rightarrow y = \exp\left(\frac{5}{3}\ln(200-3t) + C\right) = e^C(200-3t)^{5/3}$. Now $20 = y(0) = e^C \cdot 200^{5/3} \Rightarrow$ $e^C = \frac{20}{200^{5/3}}$, so $y(t) = 20\frac{(200-3t)^{5/3}}{200^{5/3}} = 20(1-0.015t)^{5/3}$ g for $0 \le t \le 66\frac{2}{3}$ s, at which time the tank is empty.

39. (a) $m\frac{dv}{dt} = mg - cv \Rightarrow \frac{dv}{dt} + \frac{c}{m}v = g$ and $I(t) = e^{\int (c/m)\,dt} = e^{(c/m)t}$, and multiplying the last differential equation by $I(t)$ gives $e^{(c/m)t}\frac{dv}{dt} + \frac{vce^{(c/m)t}}{m} = ge^{(c/m)t} \Rightarrow \left[e^{(c/m)t}v\right]' = ge^{(c/m)t}$. Hence, $v(t) = e^{-(c/m)t}\left[\int ge^{(c/m)t}\,dt + K\right] = mg/c + Ke^{-(c/m)t}$. But the object is dropped from rest, so $v(0) = 0$ and $K = -mg/c$. Thus, the velocity at time t is $v(t) = (mg/c)[1 - e^{-(c/m)t}]$.

(b) $\lim_{t\to\infty} v(t) = mg/c$