

SOLUTION TO HW #10, PART III

ANOTHER MIXING PROBLEM

Let $y = y(t)$ = the amount (lbs) of salt in the tank at time t minutes.

Let $V(t)$ = the volume (gal) of water in the tank at time t minutes.

Then, $V(0) = 100$ gallons and $y(0) = 0$ lbs. of salt.

Also, $V(t) = 100 + 5t - 3t = 100 + 2t$ gallons at time t .

Let v_{IN} = the rate (lbs/min) at which salt is entering the tank at time t .

Let v_{OUT} = the rate (lbs/min) at which salt is leaving the tank at time t .

$\frac{dy}{dt}$ = the rate (lbs/min) at which the amount of salt in the tank is changing at time t .

$$\text{Then, } \frac{dy}{dt} = v_{IN} - v_{OUT}$$

$$v_{IN} = (0.5 \text{ lbs/gal}) \times (5 \text{ gal/min}) = 2.5 \text{ lbs/min}$$

$$v_{OUT} = \left(\frac{y(t)}{V(t)} \text{ lbs/gal} \right) (3 \text{ gal/min}) = \left(\frac{3y}{100+2t} \text{ lbs/min} \right)$$

SOLUTION TO "ANOTHER MIXING PROBLEM" (p.2)

$$\text{So, } \frac{dy}{dt} = \left(2.5 - \frac{3y}{100+2t} \right) \text{ lbs/min.}$$

[Adding $\frac{3y}{100+2t}$]

$$(*) \quad \frac{dy}{dt} + \left(\frac{3}{100+2t} \right) y = 2.5 = \frac{5}{2}$$

This is a Linear First-ORDER DIFFERENTIAL EQUATION and we seek the solution y such that $y(0) = 0$.

The Integrating factor is $e^{\int \frac{3}{100+2t} dt}$.

$$\int \frac{3}{100+2t} dt = \frac{3}{2} \int \frac{1}{u} du = \frac{3}{2} \ln u$$

$$= \ln u^{3/2} = \ln((100+2t)^{3/2}),$$

$$\begin{aligned} \text{Let } u &= 100+2t \\ du &= 2dt \\ dt &= \frac{1}{2} du \end{aligned}$$

So, the Integrating Factor is $e^{\ln((100+2t)^{3/2})} = \underline{\underline{(100+2t)^{3/2}}}$.

[We multiply both sides of the differential equation at (*) above by the integrating factor.]

$$(100+2t)^{3/2} \cdot \frac{dy}{dt} + 3(100+2t)^{1/2} y = \frac{5}{2} (100+2t)^{3/2}$$

$$[(100+2t)^{3/2} y]' = \frac{5}{2} (100+2t)^{3/2}$$

$$(100+2t)^{3/2} y = \int [(100+2t)^{3/2} y]' dy = \frac{5}{2} \int (100+2t)^{3/2} dt$$

$$(100+2t)^{3/2} y = \frac{5}{2} \left(\frac{1}{2} \right) \int u^{3/2} + C$$

SOLUTION TO "ANOTHER MIXING PROBLEM" (p.3)

$$(100+2t)^{3/2} y = \frac{5}{4} \cdot \left(\frac{2}{5} u^{5/2}\right) + C$$

$$(100+2t)^{3/2} y = \frac{1}{2} (100+2t)^{5/2} + C$$

[multiplying both sides by $(100+2t)^{-3/2}$]

$$y = \frac{1}{2} (100+2t) + C (100+2t)^{-3/2}$$

[Solving for C]

When $t=0$, $y(0)=0$, so, setting $t=0$,

$$0 = \frac{1}{2} (100) + C (100)^{-3/2} = 50 + \frac{C}{100^{3/2}}$$

$$0 = 50 + \frac{C}{1000} \Rightarrow 0 = 50,000 + C$$

$\therefore C = -50,000$ and so,

$$y = 50 + t - \frac{50,000}{(100+2t)^{3/2}}$$

The tank is full when $V(t)=200$.

$$\Rightarrow 100+2t=200 \Rightarrow 2t=100 \Rightarrow t=50$$

(a) At time $t=50$ minutes, the tank is full.

$$(b) \text{ at time } t=50, y = 50 + 50 - \frac{50,000}{(100+100)^{3/2}}$$

$$y = 100 - \frac{50,000}{(200)^{3/2}} = 82.3223$$

(b) When the tank is full, the tank contains 82.32 lbs of salt.