HW #10, PART I, Sec 9.4 Solutions

3. $V(t) = \pi r^2 h(t) = 100\pi h(t) \Rightarrow \dfrac{dV}{dh} = 100\pi$ and $\dfrac{dV}{dt} = \dfrac{dV}{dh}\dfrac{dh}{dt} = 100\pi\dfrac{dh}{dt}$.

Diameter = 2.5 inches $\Rightarrow$ radius = 1.25 inches = $\frac{5}{4}\cdot\frac{1}{12}$ foot = $\frac{5}{48}$ foot. Thus, $\dfrac{dV}{dt} = -a\sqrt{2gh} \Rightarrow$

$100\pi\dfrac{dh}{dt} = -\pi\left(\frac{5}{48}\right)^2\sqrt{2\cdot 32h} = -\dfrac{25\pi}{288}\sqrt{h} \Rightarrow \dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{1152} \Rightarrow \int h^{-1/2}\,dh = \int -\frac{1}{1152}\,dt \Rightarrow$

$2\sqrt{h} = -\frac{1}{1152}t + C \Rightarrow \sqrt{h} = -\frac{1}{2304}t + k \Rightarrow h(t) = \left(-\frac{1}{2304}t + k\right)^2$. The water pressure after t seconds is $62.5h(t)$ lb/ft^2, so the condition that the pressure be at least 2160 lb/ft^2 for 10 minutes (600 seconds) is the condition $62.5\cdot h(600) \geq 2160$; that is, $\left(k - \frac{600}{2304}\right)^2 \geq \frac{2160}{62.5} \Rightarrow \left|k - \frac{25}{96}\right| \geq \sqrt{34.56} \Rightarrow k \geq \frac{25}{96} + \sqrt{34.56}$. Now $h(0) = k^2$, so the height of the tank should be at least $\left(\frac{25}{96} + \sqrt{34.56}\right)^2 \approx 37.69$ ft.

4. (a) If the radius of the circular cross-section at height h is r, then the Pythagorean Theorem gives $r^2 = 2^2 - (2-h)^2$ since the radius of the tank is 2 m. So $A(h) = \pi r^2 = \pi[4 - (2-h)^2] = \pi(4h - h^2)$. Thus, $A(h)\dfrac{dh}{dt} = -a\sqrt{2gh} \Rightarrow$

$\pi(4h - h^2)\dfrac{dh}{dt} = -\pi(0.01)^2\sqrt{2\cdot 10h} \Rightarrow (4h - h^2)\dfrac{dh}{dt} = -0.0001\sqrt{20h}$.

(b) From part (a) we have $(4h^{1/2} - h^{3/2})\,dh = \left(-0.0001\sqrt{20}\right)dt \Rightarrow \frac{8}{3}h^{3/2} - \frac{2}{5}h^{5/2} = \left(-0.0001\sqrt{20}\right)t + C$. $h(0) = 2 \Rightarrow \frac{8}{3}(2)^{3/2} - \frac{2}{5}(2)^{5/2} = C \Rightarrow C = \left(\frac{16}{3} - \frac{8}{5}\right)\sqrt{2} = \frac{56}{15}\sqrt{2}$. To find out how long it will take to drain all the water we evaluate t when $h = 0$: $0 = \left(-0.0001\sqrt{20}\right)t + C \Rightarrow$

$$t = \frac{C}{0.0001\sqrt{20}} = \frac{56\sqrt{2}/15}{0.0001\sqrt{20}} = \frac{11{,}200\sqrt{10}}{3} \approx 11{,}806 \text{ s} \approx 3 \text{ h } 17 \text{ min}$$

9.4 Models for Population Growth

1. (a) Comparing the given equation, $\dfrac{dP}{dt} = 0.04P\left(1 - \dfrac{P}{1200}\right)$, to Equation 4, $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{M}\right)$, we see that the carrying capacity is $M = 1200$ and the value of k is 0.04.

(b) By Equation 7, the solution of the equation is $P(t) = \dfrac{M}{1 + Ae^{-kt}}$, where $A = \dfrac{M - P_0}{P_0}$. Since $P(0) = P_0 = 60$, we have $A = \dfrac{1200 - 60}{60} = 19$, and hence, $P(t) = \dfrac{1200}{1 + 19e^{-0.04t}}$.

(c) The population after 10 weeks is $P(10) = \dfrac{1200}{1 + 19e^{-0.04(10)}} \approx 87$.

2. (a) $dP/dt = 0.02P - 0.0004P^2 = 0.02P(1 - 0.02P) = 0.02P(1 - P/50)$. Comparing to Equation 4, $dP/dt = kP(1 - P/M)$, we see that the carrying capacity is $M = 50$ and the value of k is 0.02.

(b) By Equation 7, the solution of the equation is $P(t) = \dfrac{M}{1 + Ae^{-kt}}$, where $A = \dfrac{M - P_0}{P_0}$. Since $P(0) = P_0 = 40$, we have $A = \dfrac{50 - 40}{40} = 0.25$, and hence, $P(t) = \dfrac{50}{1 + 0.25e^{-0.02t}}$.

(c) The population after 10 weeks is $P(10) = \dfrac{50}{1 + 0.25e^{-0.02(10)}} \approx 42$.

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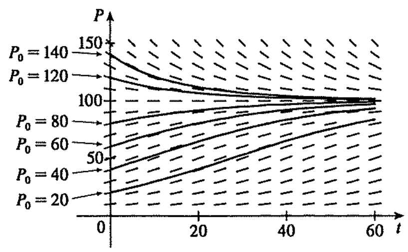
3. (a) $dP/dt = 0.05P - 0.0005P^2 = 0.05P(1 - 0.01P) = 0.05P(1 - P/100)$. Comparing to Equation 4, $dP/dt = kP(1 - P/M)$, we see that the carrying capacity is $M = 100$ and the value of k is 0.05.

(b) The slopes close to 0 occur where P is near 0 or 100. The largest slopes appear to be on the line $P = 50$. The solutions are increasing for $0 < P_0 < 100$ and decreasing for $P_0 > 100$.

(c)

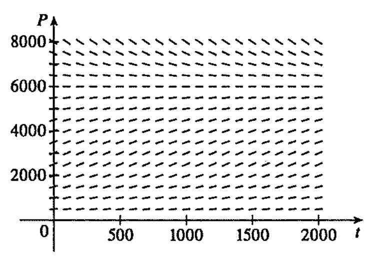


All of the solutions approach $P = 100$ as t increases. As in part (b), the solutions differ since for $0 < P_0 < 100$ they are increasing, and for $P_0 > 100$ they are decreasing. Also, some have an IP and some don't. It appears that the solutions which have $P_0 = 20$ and $P_0 = 40$ have inflection points at $P = 50$.

(d) The equilibrium solutions are $P = 0$ (trivial solution) and $P = 100$. The increasing solutions move away from $P = 0$ and all nonzero solutions approach $P = 100$ as $t \to \infty$.

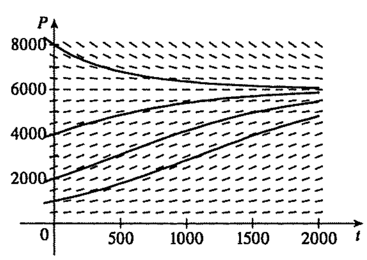
4. (a) $M = 6000$ and $k = 0.0015 \Rightarrow dP/dt = 0.0015P(1 - P/6000)$.

(b)



All of the solution curves approach 6000 as $t \to \infty$.

(c)

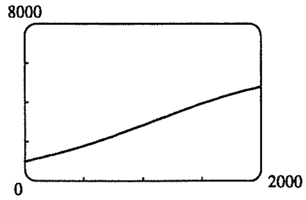


The curves with $P_0 = 1000$ and $P_0 = 2000$ appear to be concave upward at first and then concave downward. The curve with $P_0 = 4000$ appears to be concave downward everywhere. The curve with $P_0 = 8000$ appears to be concave upward everywhere. The inflection points are where the population grows the fastest.

(d) See the solution to Exercise 9.2.25 for a possible program to calculate $P(50)$. [In this case, we use $\mathrm{X} = 0$, $\mathrm{H} = 1$, $\mathrm{N} = 50$, $\mathrm{Y}_1 = 0.0015y(1 - y/6000)$, and $\mathrm{Y} = 1000$.] We find that $P(50) \approx 1064$.

(e) Using Equation 7 with $M = 6000$, $k = 0.0015$, and $P_0 = 1000$, we have $P(t) = \dfrac{M}{1 + Ae^{-kt}} = \dfrac{6000}{1 + Ae^{-0.0015t}}$, where $A = \dfrac{M - P_0}{P_0} = \dfrac{6000 - 1000}{1000} = 5$. Thus, $P(50) = \dfrac{6000}{1 + 5e^{-0.0015(50)}} \approx 1064.1$, which is extremely close to the estimate obtained in part (d).

(f)



The curves are very similar.

5. (a) $\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right) \Rightarrow y(t) = \frac{M}{1 + Ae^{-kt}}$ with $A = \frac{M - y(0)}{y(0)}$. With $M = 8 \times 10^7$, $k = 0.71$, and $y(0) = 2 \times 10^7$, we get the model $y(t) = \frac{8 \times 10^7}{1 + 3e^{-0.71t}}$, so $y(1) = \frac{8 \times 10^7}{1 + 3e^{-0.71}} \approx 3.23 \times 10^7$ kg.

(b) $y(t) = 4 \times 10^7 \Rightarrow \frac{8 \times 10^7}{1 + 3e^{-0.71t}} = 4 \times 10^7 \Rightarrow 2 = 1 + 3e^{-0.71t} \Rightarrow e^{-0.71t} = \frac{1}{3} \Rightarrow$
$-0.71t = \ln\frac{1}{3} \Rightarrow t = \frac{\ln 3}{0.71} \approx 1.55$ years

6. (a) $\frac{dP}{dt} = 0.4P - 0.001P^2 = 0.4P(1 - 0.0025P)$ $\left[\frac{0.001}{0.4} = 0.0025\right]$ $= 0.4P\left(1 - \frac{P}{400}\right)$ $\left[0.0025^{-1} = 400\right]$

Thus, by Equation 4, $k = 0.4$ and the carrying capacity is 400.

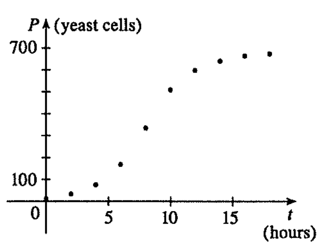
(b) Using the fact that $P(0) = 50$ and the formula for dP/dt, we get

$$P'(0) = \left.\frac{dP}{dt}\right|_{t=0} = 0.4(50) - 0.001(50)^2 = 20 - 2.5 = 17.5.$$

(c) From Equation 7, $A = \frac{M - P_0}{P_0} = \frac{400 - 50}{50} = 7$, so $P = \frac{400}{1 + 7e^{-0.4t}}$. The population reaches 50% of the carrying capacity, 200, when $200 = \frac{400}{1 + 7e^{-0.4t}} \Rightarrow 1 + 7e^{-0.4t} = 2 \Rightarrow e^{-0.4t} = \frac{1}{7} \Rightarrow -0.4t = \ln\frac{1}{7} \Rightarrow$
$t = \left(\ln\frac{1}{7}\right)/(-0.4) \approx 4.86$ years.

7. Using Equation 7, $A = \frac{M - P_0}{P_0} = \frac{10{,}000 - 1000}{1000} = 9$, so $P(t) = \frac{10{,}000}{1 + 9e^{-kt}}$. $P(1) = 2500 \Rightarrow$
$2500 = \frac{10{,}000}{1 + 9e^{-k(1)}} \Rightarrow 1 + 9e^{-k} = 4 \Rightarrow 9e^{-k} = 3 \Rightarrow e^{-k} = \frac{1}{3} \Rightarrow -k = \ln\frac{1}{3} \Rightarrow k = \ln 3$. After another three years, $t = 4$, and $P(4) = \frac{10{,}000}{1 + 9e^{-(\ln 3)4}} = \frac{10{,}000}{1 + 9\left(e^{\ln 3}\right)^{-4}} = \frac{10{,}000}{1 + 9(3)^{-4}} = \frac{10{,}000}{1 + \frac{1}{9}} = \frac{10{,}000}{\frac{10}{9}} = 9000.$

8. (a)



From the graph, we estimate the carrying capacity M for the yeast population to be 680.

(b) An estimate of the initial relative growth rate is $\frac{1}{P_0}\frac{dP}{dt} = \frac{1}{18} \cdot \frac{39 - 18}{2 - 0} = \frac{7}{12} = 0.58\overline{3}$.

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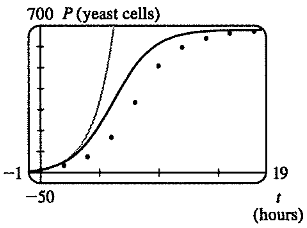
(c) An exponential model is $P(t) = 18e^{7t/12}$. A logistic model is $P(t) = \dfrac{680}{1 + Ae^{-7t/12}}$, where $A = \frac{680-18}{18} = \frac{331}{9}$.

(d)

Time in Hours	Observed Values	Exponential Model	Logistic Model
0	18	18	18
2	39	58	55
4	80	186	149
6	171	596	322
8	336	1914	505
10	509	6147	614
12	597	19,739	658
14	640	63,389	673
16	664	203,558	678
18	672	653,679	679



The exponential model is a poor fit for anything beyond the first two observed values. The logistic model varies more for the middle values than it does for the values at either end, but provides a good general fit, as shown in the figure.

(e) $P(7) = \dfrac{680}{1 + \frac{331}{9} e^{-7(7/12)}} \approx 420$ yeast cells

9. (a) We will assume that the difference in birth and death rates is 20 million/year. Let $t = 0$ correspond to the year 2000. Thus,

$$k \approx \frac{1}{P}\frac{dP}{dt} = \frac{1}{6.1 \text{ billion}}\left(\frac{20 \text{ million}}{\text{year}}\right) = \frac{1}{305}, \text{ and } \frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) = \frac{1}{305}P\left(1 - \frac{P}{20}\right) \text{ with } P \text{ in billions.}$$

(b) $A = \dfrac{M - P_0}{P_0} = \dfrac{20 - 6.1}{6.1} = \dfrac{139}{61} \approx 2.2787$. $P(t) = \dfrac{M}{1 + Ae^{-kt}} = \dfrac{20}{1 + \frac{139}{61}e^{-t/305}}$, so

$P(10) = \dfrac{20}{1 + \frac{139}{61}e^{-10/305}} \approx 6.24$ billion, which underestimates the actual 2010 population of 6.9 billion.

(c) The years 2100 and 2500 correspond to $t = 100$ and $t = 500$, respectively. $P(100) = \dfrac{20}{1 + \frac{139}{61}e^{-100/305}} \approx 7.57$ billion

and $P(500) = \dfrac{20}{1 + \frac{139}{61}e^{-500/305}} \approx 13.87$ billion.

10. (a) Let $t = 0$ correspond to the year 2000. $A = \dfrac{M - P_0}{P_0} = \dfrac{800 - 282}{282} = \dfrac{259}{141} \approx 1.8369$.

$P(t) = \dfrac{M}{1 + Ae^{-kt}} = \dfrac{800}{1 + \frac{259}{141}e^{-kt}}$ with P in millions.

(b) $P(10) = 309 \quad\Leftrightarrow\quad \dfrac{800}{1 + \frac{259}{141}e^{-10k}} = 309 \quad\Leftrightarrow\quad \dfrac{800}{309} = 1 + \dfrac{259}{141}e^{-10k} \quad\Leftrightarrow\quad \dfrac{491}{309} = \dfrac{259}{141}e^{-10k} \quad\Leftrightarrow$

$\dfrac{491 \cdot 141}{309 \cdot 259} = e^{-10k} \quad\Leftrightarrow\quad -10k = \ln\dfrac{491 \cdot 47}{103 \cdot 259} \quad\Leftrightarrow\quad k = -\dfrac{1}{10}\ln\dfrac{23{,}077}{26{,}677} \approx 0.0145.$

(c) The years 2100 and 2200 correspond to $t = 100$ and $t = 200$, respectively. $P(100) = \dfrac{800}{1 + \frac{259}{141}e^{-100k}} \approx 559$ million and

$P(200) = \dfrac{800}{1 + \frac{259}{141}e^{-200k}} \approx 727$ million.

(d) $P(t)=500 \Leftrightarrow \dfrac{800}{1+\frac{259}{141}e^{-kt}}=500 \Leftrightarrow \dfrac{800}{500}=1+\dfrac{259}{141}e^{-kt} \Leftrightarrow \dfrac{3}{5}=\dfrac{259}{141}e^{-kt} \Leftrightarrow \dfrac{3\cdot 141}{5\cdot 259}=e^{-kt} \Leftrightarrow$

$-kt=\ln\dfrac{423}{1295} \Leftrightarrow t=10\dfrac{\ln(423/1295)}{\ln(23{,}077/26{,}677)}\approx 77.18$ years. Our logistic model predicts that the US population will exceed 500 million in 77.18 years; that is, in the year 2077.

11. (a) Our assumption is that $\dfrac{dy}{dt}=ky(1-y)$, where y is the fraction of the population that has heard the rumor.

(b) Using the logistic equation (4), $\dfrac{dP}{dt}=kP\left(1-\dfrac{P}{M}\right)$, we substitute $y=\dfrac{P}{M}$, $P=My$, and $\dfrac{dP}{dt}=M\,\dfrac{dy}{dt}$,

to obtain $M\,\dfrac{dy}{dt}=k(My)(1-y) \Leftrightarrow \dfrac{dy}{dt}=ky(1-y)$, our equation in part (a).

Now the solution to (4) is $P(t)=\dfrac{M}{1+Ae^{-kt}}$, where $A=\dfrac{M-P_0}{P_0}$.

We use the same substitution to obtain $My=\dfrac{M}{1+\dfrac{M-My_0}{My_0}e^{-kt}} \Rightarrow y=\dfrac{y_0}{y_0+(1-y_0)e^{-kt}}$.

Alternatively, we could use the same steps as outlined in the solution of Equation 4.

(c) Let t be the number of hours since 8 AM. Then $y_0=y(0)=\frac{80}{1000}=0.08$ and $y(4)=\frac{1}{2}$, so

$\dfrac{1}{2}=y(4)=\dfrac{0.08}{0.08+0.92e^{-4k}}$. Thus, $0.08+0.92e^{-4k}=0.16$, $e^{-4k}=\frac{0.08}{0.92}=\frac{2}{23}$, and $e^{-k}=\left(\frac{2}{23}\right)^{1/4}$,

so $y=\dfrac{0.08}{0.08+0.92(2/23)^{t/4}}=\dfrac{2}{2+23(2/23)^{t/4}}$. Solving this equation for t, we get

$$2y+23y\left(\frac{2}{23}\right)^{t/4}=2 \Rightarrow \left(\frac{2}{23}\right)^{t/4}=\frac{2-2y}{23y} \Rightarrow \left(\frac{2}{23}\right)^{t/4}=\frac{2}{23}\cdot\frac{1-y}{y} \Rightarrow \left(\frac{2}{23}\right)^{t/4-1}=\frac{1-y}{y}.$$

It follows that $\dfrac{t}{4}-1=\dfrac{\ln[(1-y)/y]}{\ln\frac{2}{23}}$, so $t=4\left[1+\dfrac{\ln((1-y)/y)}{\ln\frac{2}{23}}\right]$.

When $y=0.9$, $\dfrac{1-y}{y}=\frac{1}{9}$, so $t=4\left(1-\dfrac{\ln 9}{\ln\frac{2}{23}}\right)\approx 7.6$ h or 7 h 36 min. Thus, 90% of the population will have heard the rumor by 3:36 PM.

12. (a) $P(0)=P_0=400$, $P(1)=1200$ and $M=10{,}000$. From the solution to the logistic differential equation

$P(t)=\dfrac{P_0M}{P_0+(M-P_0)e^{-kt}}$, we get $P=\dfrac{400\,(10{,}000)}{400+(9600)e^{-kt}}=\dfrac{10{,}000}{1+24e^{-kt}}$. $P(1)=1200 \Rightarrow$

$1+24e^{-k}=\frac{100}{12} \Rightarrow e^k=\frac{288}{88} \Rightarrow k=\ln\frac{36}{11}$. So $P=\dfrac{10{,}000}{1+24e^{-t\ln(36/11)}}=\dfrac{10{,}000}{1+24\cdot(11/36)^t}$.

(b) $5000=\dfrac{10{,}000}{1+24(11/36)^t} \Rightarrow 24\left(\frac{11}{36}\right)^t=1 \Rightarrow t\ln\frac{11}{36}=\ln\frac{1}{24} \Rightarrow t\approx 2.68$ years.

13. (a) $\dfrac{dP}{dt}=kP\left(1-\dfrac{P}{M}\right) \Rightarrow \dfrac{d^2P}{dt^2}=k\left[P\left(-\dfrac{1}{M}\dfrac{dP}{dt}\right)+\left(1-\dfrac{P}{M}\right)\dfrac{dP}{dt}\right]=k\,\dfrac{dP}{dt}\left(-\dfrac{P}{M}+1-\dfrac{P}{M}\right)$

$$=k\left[kP\left(1-\frac{P}{M}\right)\right]\left(1-\frac{2P}{M}\right)=k^2P\left(1-\frac{P}{M}\right)\left(1-\frac{2P}{M}\right)$$