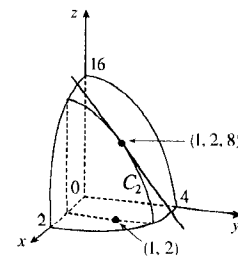
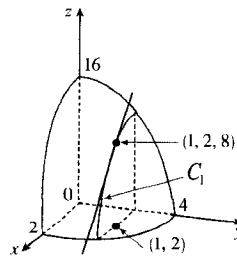


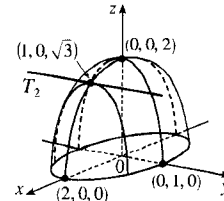
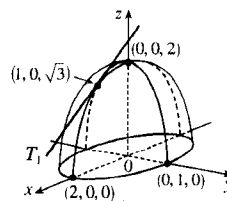
HW # 11, Sec 14.3 Solutions

Sec 14.3

(the curve C_1 in the first figure). The slope of the tangent line to this parabola at $(1, 2, 8)$ is $f_x(1, 2) = -8$. Similarly the plane $x = 1$ intersects the paraboloid in the parabola $z = 12 - y^2$, $x = 1$ (the curve C_2 in the second figure) and the slope of the tangent line at $(1, 2, 8)$ is $f_y(1, 2) = -4$.



8. $f(x, y) = (4 - x^2 - 4y^2)^{1/2} \Rightarrow f_x(x, y) = -x(4 - x^2 - 4y^2)^{-1/2}$ and $f_y(x, y) = -4y(4 - x^2 - 4y^2)^{-1/2} \Rightarrow f_x(1, 0) = -\frac{1}{\sqrt{3}}$, $f_y(1, 0) = 0$. The graph of f is the upper half of the ellipsoid $z^2 + x^2 + 4y^2 = 4$ and the plane $y = 0$ intersects the graph in the semicircle $x^2 + z^2 = 4$, $z \geq 0$ and the slope of the tangent line T_1 to this semicircle at $(1, 0, \sqrt{3})$ is $f_x(1, 0) = -\frac{1}{\sqrt{3}}$. Similarly the plane $x = 1$ intersects the graph in the semi-ellipse $z^2 + 4y^2 = 3$, $z \geq 0$ and the slope of the tangent line T_2 to this semi-ellipse at $(1, 0, \sqrt{3})$ is $f_y(1, 0) = 0$.



9. $f(x, y) = x^4 + 5xy^3 \Rightarrow f_x(x, y) = 4x^3 + 5y^3$, $f_y(x, y) = 0 + 5x \cdot 3y^2 = 15xy^2$
10. $f(x, y) = x^2y - 3y^4 \Rightarrow f_x(x, y) = 2x \cdot y - 0 = 2xy$, $f_y(x, y) = x^2 \cdot 1 - 3 \cdot 4y^3 = x^2 - 12y^3$
11. $g(x, y) = x^3 \sin y \Rightarrow g_x(x, y) = 3x^2 \sin y$, $g_y(x, y) = x^3 \cos y$
12. $g(x, t) = e^{xt} \Rightarrow g_x(x, t) = e^{xt} \cdot t = te^{xt}$, $g_t(x, t) = e^{xt} \cdot x = xe^{xt}$
13. $z = \ln(x + t^2) \Rightarrow \frac{\partial z}{\partial x} = \frac{1}{x + t^2} \cdot 1 = \frac{1}{x + t^2}$, $\frac{\partial z}{\partial t} = \frac{1}{x + t^2} \cdot 2t = \frac{2t}{x + t^2}$
14. $w = \frac{u}{v^2} \Rightarrow w_u = \frac{1}{v^2} \cdot 1 = \frac{1}{v^2}$, $w_v = u \cdot (-2v^{-3}) = -\frac{2u}{v^3}$
15. $f(x, y) = ye^{xy} \Rightarrow f_x(x, y) = y \cdot e^{xy} \cdot y = y^2e^{xy}$, $f_y(x, y) = y \cdot e^{xy} \cdot x + e^{xy} \cdot 1 = e^{xy} + xye^{xy}$
16. $g(x, y) = (x^2 + xy)^3 \Rightarrow g_x(x, y) = 3(x^2 + xy)^2(2x + y)$, $g_y(x, y) = 3(x^2 + xy)^2(0 + x) = 3x(x^2 + xy)^2$
17. $g(x, y) = y(x + x^2y)^5 \Rightarrow g_x(x, y) = 5y(x + x^2y)^4(1 + 2xy)$,
 $g_y(x, y) = y \cdot 5(x + x^2y)^4 \cdot x^2 + (x + x^2y)^5 \cdot 1 = 5x^2y(x + x^2y)^4 + (x + x^2y)^5$
18. $f(x, y) = \frac{x}{(x + y)^2} \Rightarrow f_x(x, y) = \frac{(x + y)^2(1) - (x)(2)(x + y)}{[(x + y)^2]^2} = \frac{x + y - 2x}{(x + y)^3} = \frac{y - x}{(x + y)^3}$,
 $f_y(x, y) = \frac{(x + y)^2(0) - (x)(2)(x + y)}{[(x + y)^2]^2} = -\frac{2x}{(x + y)^3}$

19. $f(x, y) = \frac{ax + by}{cx + dy} \Rightarrow f_x(x, y) = \frac{(cx + dy)(a) - (ax + by)(c)}{(cx + dy)^2} = \frac{(ad - bc)y}{(cx + dy)^2},$
 $f_y(x, y) = \frac{(cx + dy)(b) - (ax + by)(d)}{(cx + dy)^2} = \frac{(bc - ad)x}{(cx + dy)^2}$
20. $w = \frac{e^v}{u + v^2} \Rightarrow \frac{\partial w}{\partial u} = \frac{0(u + v^2) - e^v(1)}{(u + v^2)^2} = -\frac{e^v}{(u + v^2)^2}, \quad \frac{\partial w}{\partial v} = \frac{e^v(u + v^2) - e^v(2v)}{(u + v^2)^2} = \frac{e^v(u + v^2 - 2v)}{(u + v^2)^2}$
21. $g(u, v) = (u^2v - v^3)^5 \Rightarrow g_u(u, v) = 5(u^2v - v^3)^4 \cdot 2uv = 10uv(u^2v - v^3)^4,$
 $g_v(u, v) = 5(u^2v - v^3)^4(u^2 - 3v^2) = 5(u^2 - 3v^2)(u^2v - v^3)^4$
22. $u(r, \theta) = \sin(r \cos \theta) \Rightarrow u_r(r, \theta) = \cos(r \cos \theta) \cdot \cos \theta = \cos \theta \cos(r \cos \theta),$
 $u_\theta(r, \theta) = \cos(r \cos \theta)(-r \sin \theta) = -r \sin \theta \cos(r \cos \theta)$
23. $R(p, q) = \tan^{-1}(pq^2) \Rightarrow R_p(p, q) = \frac{1}{1 + (pq^2)^2} \cdot q^2 = \frac{q^2}{1 + p^2q^4}, \quad R_q(p, q) = \frac{1}{1 + (pq^2)^2} \cdot 2pq = \frac{2pq}{1 + p^2q^4}$
24. $f(x, y) = x^y \Rightarrow f_x(x, y) = yx^{y-1}, \quad f_y(x, y) = x^y \ln x$
25. $F(x, y) = \int_y^x \cos(e^t) dt \Rightarrow F_x(x, y) = \frac{\partial}{\partial x} \int_y^x \cos(e^t) dt = \cos(e^x)$ by the Fundamental Theorem of Calculus, Part 1;
 $F_y(x, y) = \frac{\partial}{\partial y} \int_y^x \cos(e^t) dt = \frac{\partial}{\partial y} \left[-\int_x^y \cos(e^t) dt \right] = -\frac{\partial}{\partial y} \int_x^y \cos(e^t) dt = -\cos(e^y).$
26. $F(\alpha, \beta) = \int_\alpha^\beta \sqrt{t^3 + 1} dt \Rightarrow$
 $F_\alpha(\alpha, \beta) = \frac{\partial}{\partial \alpha} \int_\alpha^\beta \sqrt{t^3 + 1} dt = \frac{\partial}{\partial \alpha} \left[-\int_\beta^\alpha \sqrt{t^3 + 1} dt \right] = -\frac{\partial}{\partial \alpha} \int_\beta^\alpha \sqrt{t^3 + 1} dt = -\sqrt{\alpha^3 + 1}$ by the Fundamental Theorem of Calculus, Part 1;
 $F_\beta(\alpha, \beta) = \frac{\partial}{\partial \beta} \int_\alpha^\beta \sqrt{t^3 + 1} dt = \sqrt{\beta^3 + 1}.$
27. $f(x, y, z) = x^3yz^2 + 2yz \Rightarrow f_x(x, y, z) = 3x^2yz^2, \quad f_y(x, y, z) = x^3z^2 + 2z, \quad f_z(x, y, z) = 2x^3yz + 2y$
28. $f(x, y, z) = xy^2e^{-xz} \Rightarrow f_x(x, y, z) = y^2[x \cdot e^{-xz}(-z) + e^{-xz} \cdot 1] = (1 - xz)y^2e^{-xz}, \quad f_y(x, y, z) = 2xye^{-xz},$
 $f_z(x, y, z) = xy^2e^{-xz}(-x) = -x^2y^2e^{-xz}$
29. $w = \ln(x + 2y + 3z) \Rightarrow \frac{\partial w}{\partial x} = \frac{1}{x + 2y + 3z}, \quad \frac{\partial w}{\partial y} = \frac{2}{x + 2y + 3z}, \quad \frac{\partial w}{\partial z} = \frac{3}{x + 2y + 3z}$
30. $w = y \tan(x + 2z) \Rightarrow \frac{\partial w}{\partial x} = y [\sec^2(x + 2z)](1) = y \sec^2(x + 2z), \quad \frac{\partial w}{\partial y} = \tan(x + 2z),$
 $\frac{\partial w}{\partial z} = y [\sec^2(x + 2z)](2) = 2y \sec^2(x + 2z)$
31. $p = \sqrt{t^4 + u^2 \cos v} \Rightarrow \frac{\partial p}{\partial t} = \frac{1}{2}(t^4 + u^2 \cos v)^{-1/2}(4t^3) = \frac{2t^3}{\sqrt{t^4 + u^2 \cos v}},$
 $\frac{\partial p}{\partial u} = \frac{1}{2}(t^4 + u^2 \cos v)^{-1/2}(2u \cos v) = \frac{u \cos v}{\sqrt{t^4 + u^2 \cos v}}, \quad \frac{\partial p}{\partial v} = \frac{1}{2}(t^4 + u^2 \cos v)^{-1/2}[u^2(-\sin v)] = -\frac{u^2 \sin v}{2\sqrt{t^4 + u^2 \cos v}}$

$$32. u = x^{y/z} \Rightarrow u_x = \frac{y}{z} x^{(y/z)-1}, u_y = x^{y/z} \ln x \cdot \frac{1}{z} = \frac{x^{y/z}}{z} \ln x, u_z = x^{y/z} \ln x \cdot \frac{-y}{z^2} = -\frac{yx^{y/z}}{z^2} \ln x$$

$$33. h(x, y, z, t) = x^2 y \cos(z/t) \Rightarrow h_x(x, y, z, t) = 2xy \cos(z/t), h_y(x, y, z, t) = x^2 \cos(z/t), \\ h_z(x, y, z, t) = -x^2 y \sin(z/t)(1/t) = (-x^2 y/t) \sin(z/t), h_t(x, y, z, t) = -x^2 y \sin(z/t)(-zt^{-2}) = (x^2 y z/t^2) \sin(z/t)$$

$$34. \phi(x, y, z, t) = \frac{\alpha x + \beta y^2}{\gamma z + \delta t^2} \Rightarrow \phi_x(x, y, z, t) = \frac{1}{\gamma z + \delta t^2} (\alpha) = \frac{\alpha}{\gamma z + \delta t^2}, \\ \phi_y(x, y, z, t) = \frac{1}{\gamma z + \delta t^2} (2\beta y) = \frac{2\beta y}{\gamma z + \delta t^2}, \phi_z(x, y, z, t) = \frac{(\gamma z + \delta t^2)(0) - (\alpha x + \beta y^2)(\gamma)}{(\gamma z + \delta t^2)^2} = \frac{-\gamma(\alpha x + \beta y^2)}{(\gamma z + \delta t^2)^2}, \\ \phi_t(x, y, z, t) = \frac{(\gamma z + \delta t^2)(0) - (\alpha x + \beta y^2)(2\delta t)}{(\gamma z + \delta t^2)^2} = -\frac{2\delta t(\alpha x + \beta y^2)}{(\gamma z + \delta t^2)^2}$$

$$35. u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \text{ For each } i = 1, \dots, n, u_{x_i} = \frac{1}{2} (x_1^2 + x_2^2 + \cdots + x_n^2)^{-1/2} (2x_i) = \frac{x_i}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}.$$

$$36. u = \sin(x_1 + 2x_2 + \cdots + nx_n). \text{ For each } i = 1, \dots, n, u_{x_i} = i \cos(x_1 + 2x_2 + \cdots + nx_n).$$

$$(37) R(s, t) = te^{s/t} \Rightarrow R_t(s, t) = t \cdot e^{s/t} (-s/t^2) + e^{s/t} \cdot 1 = \left(1 - \frac{s}{t}\right) e^{s/t}, \text{ so } R_t(0, 1) = \left(1 - \frac{0}{1}\right) e^{0/1} = 1.$$

$$38. f(x, y) = y \sin^{-1}(xy) \Rightarrow f_y(x, y) = y \cdot \frac{1}{\sqrt{1 - (xy)^2}} (x) + \sin^{-1}(xy) \cdot 1 = \frac{xy}{\sqrt{1 - x^2 y^2}} + \sin^{-1}(xy),$$

$$\text{so } f_y(1, \frac{1}{2}) = \frac{1 \cdot \frac{1}{2}}{\sqrt{1 - 1^2 (\frac{1}{2})^2}} + \sin^{-1}(1 \cdot \frac{1}{2}) = \frac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} + \sin^{-1} \frac{1}{2} = \frac{1}{\sqrt{3}} + \frac{\pi}{6}.$$

$$39. f(x, y, z) = \ln \frac{1 - \sqrt{x^2 + y^2 + z^2}}{1 + \sqrt{x^2 + y^2 + z^2}} \Rightarrow$$

$$f_y(x, y, z) = \frac{1}{\frac{1 - \sqrt{x^2 + y^2 + z^2}}{1 + \sqrt{x^2 + y^2 + z^2}}} \cdot \frac{\left(1 + \sqrt{x^2 + y^2 + z^2}\right) \left(-\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2y\right) - \left(1 - \sqrt{x^2 + y^2 + z^2}\right) \left(\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2y\right)}{\left(1 + \sqrt{x^2 + y^2 + z^2}\right)^2} \\ = \frac{1 + \sqrt{x^2 + y^2 + z^2}}{1 - \sqrt{x^2 + y^2 + z^2}} \cdot \frac{-y(x^2 + y^2 + z^2)^{-1/2} \left(1 + \sqrt{x^2 + y^2 + z^2} + 1 - \sqrt{x^2 + y^2 + z^2}\right)}{\left(1 + \sqrt{x^2 + y^2 + z^2}\right)^2} \\ = \frac{-y(x^2 + y^2 + z^2)^{-1/2} (2)}{\left(1 - \sqrt{x^2 + y^2 + z^2}\right) \left(1 + \sqrt{x^2 + y^2 + z^2}\right)} = \frac{-2y}{\sqrt{x^2 + y^2 + z^2} [1 - (x^2 + y^2 + z^2)]} \\ \text{so } f_y(1, 2, 2) = \frac{-2(2)}{\sqrt{1^2 + 2^2 + 2^2} [1 - (1^2 + 2^2 + 2^2)]} = \frac{-4}{\sqrt{9} (1 - 9)} = \frac{1}{6}.$$

$$40. f(x, y, z) = x^{yz} \Rightarrow f_z(x, y, z) = (x^{yz} \ln x)(y) = yx^{yz} \ln x, \text{ so } f_z(e, 1, 0) = 1e^{(1)(0)} \ln e = 1.$$

$$41. x^2 + 2y^2 + 3z^2 = 1 \Rightarrow \frac{\partial}{\partial x}(x^2 + 2y^2 + 3z^2) = \frac{\partial}{\partial x}(1) \Rightarrow 2x + 0 + 6z \frac{\partial z}{\partial x} = 0 \Rightarrow 6z \frac{\partial z}{\partial x} = -2x \Rightarrow$$

$$\frac{\partial z}{\partial x} = \frac{-2x}{6z} = -\frac{x}{3z}, \text{ and } \frac{\partial}{\partial y}(x^2 + 2y^2 + 3z^2) = \frac{\partial}{\partial y}(1) \Rightarrow 0 + 4y + 6z \frac{\partial z}{\partial y} = 0 \Rightarrow 6z \frac{\partial z}{\partial y} = -4y \Rightarrow$$

$$\frac{\partial z}{\partial y} = \frac{-4y}{6z} = -\frac{2y}{3z}.$$

$$42. x^2 - y^2 + z^2 - 2z = 4 \Rightarrow \frac{\partial}{\partial x}(x^2 - y^2 + z^2 - 2z) = \frac{\partial}{\partial x}(4) \Rightarrow 2x - 0 + 2z \frac{\partial z}{\partial x} - 2 \frac{\partial z}{\partial x} = 0 \Rightarrow$$

$$(2z - 2) \frac{\partial z}{\partial x} = -2x \Rightarrow \frac{\partial z}{\partial x} = \frac{-2x}{2z - 2} = \frac{x}{1 - z}, \text{ and } \frac{\partial}{\partial y}(x^2 - y^2 + z^2 - 2z) = \frac{\partial}{\partial y}(4) \Rightarrow$$

$$0 - 2y + 2z \frac{\partial z}{\partial y} - 2 \frac{\partial z}{\partial y} = 0 \Rightarrow (2z - 2) \frac{\partial z}{\partial y} = 2y \Rightarrow \frac{\partial z}{\partial y} = \frac{2y}{2z - 2} = \frac{y}{z - 1}.$$

$$43. e^z = xyz \Rightarrow \frac{\partial}{\partial x}(e^z) = \frac{\partial}{\partial x}(xyz) \Rightarrow e^z \frac{\partial z}{\partial x} = y \left(x \frac{\partial z}{\partial x} + z \cdot 1 \right) \Rightarrow e^z \frac{\partial z}{\partial x} - xy \frac{\partial z}{\partial x} = yz \Rightarrow$$

$$(e^z - xy) \frac{\partial z}{\partial x} = yz, \text{ so } \frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}.$$

$$\frac{\partial}{\partial y}(e^z) = \frac{\partial}{\partial y}(xyz) \Rightarrow e^z \frac{\partial z}{\partial y} = x \left(y \frac{\partial z}{\partial y} + z \cdot 1 \right) \Rightarrow e^z \frac{\partial z}{\partial y} - xy \frac{\partial z}{\partial y} = xz \Rightarrow (e^z - xy) \frac{\partial z}{\partial y} = xz, \text{ so}$$

$$\frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}.$$

$$44. yz + x \ln y = z^2 \Rightarrow \frac{\partial}{\partial x}(yz + x \ln y) = \frac{\partial}{\partial x}(z^2) \Rightarrow y \frac{\partial z}{\partial x} + \ln y = 2z \frac{\partial z}{\partial x} \Rightarrow \ln y = 2z \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial x} \Rightarrow$$

$$\ln y = (2z - y) \frac{\partial z}{\partial x}, \text{ so } \frac{\partial z}{\partial x} = \frac{\ln y}{2z - y}.$$

$$\frac{\partial}{\partial y}(yz + x \ln y) = \frac{\partial}{\partial y}(z^2) \Rightarrow y \frac{\partial z}{\partial y} + z \cdot 1 + x \cdot \frac{1}{y} = 2z \frac{\partial z}{\partial y} \Rightarrow z + \frac{x}{y} = 2z \frac{\partial z}{\partial y} - y \frac{\partial z}{\partial y} \Rightarrow$$

$$z + \frac{x}{y} = (2z - y) \frac{\partial z}{\partial y}, \text{ so } \frac{\partial z}{\partial y} = \frac{z + (x/y)}{2z - y} = \frac{x + yz}{y(2z - y)}.$$

$$45. (a) z = f(x) + g(y) \Rightarrow \frac{\partial z}{\partial x} = f'(x), \quad \frac{\partial z}{\partial y} = g'(y)$$

$$(b) z = f(x + y). \text{ Let } u = x + y. \text{ Then } \frac{\partial z}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x} = \frac{df}{du}(1) = f'(u) = f'(x + y),$$

$$\frac{\partial z}{\partial y} = \frac{df}{du} \frac{\partial u}{\partial y} = \frac{df}{du}(1) = f'(u) = f'(x + y).$$

46. (a) $z = f(x)g(y) \Rightarrow \frac{\partial z}{\partial x} = f'(x)g(y), \quad \frac{\partial z}{\partial y} = f(x)g'(y)$

(b) $z = f(xy)$. Let $u = xy$. Then $\frac{\partial u}{\partial x} = y$ and $\frac{\partial u}{\partial y} = x$. Hence $\frac{\partial z}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x} = \frac{df}{du} \cdot y = yf'(u) = yf'(xy)$

and $\frac{\partial z}{\partial y} = \frac{df}{du} \frac{\partial u}{\partial y} = \frac{df}{du} \cdot x = xf'(u) = xf'(xy)$.

(c) $z = f\left(\frac{x}{y}\right)$. Let $u = \frac{x}{y}$. Then $\frac{\partial u}{\partial x} = \frac{1}{y}$ and $\frac{\partial u}{\partial y} = -\frac{x}{y^2}$. Hence $\frac{\partial z}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x} = f'(u) \frac{1}{y} = \frac{f'(x/y)}{y}$

and $\frac{\partial z}{\partial y} = \frac{df}{du} \frac{\partial u}{\partial y} = f'(u) \left(-\frac{x}{y^2}\right) = -\frac{xf'(x/y)}{y^2}$.

47. $f(x, y) = x^4y - 2x^3y^2 \Rightarrow f_x(x, y) = 4x^3y - 6x^2y^2, \quad f_y(x, y) = x^4 - 4x^3y$. Then $f_{xx}(x, y) = 12x^2y - 12xy^2$,
 $f_{xy}(x, y) = 4x^3 - 12x^2y, \quad f_{yx}(x, y) = 4x^3 - 12x^2y$, and $f_{yy}(x, y) = -4x^3$.

48. $f(x, y) = \ln(ax + by) \Rightarrow f_x(x, y) = \frac{a}{ax + by} = a(ax + by)^{-1}, \quad f_y(x, y) = \frac{b}{ax + by} = b(ax + by)^{-1}$. Then

$f_{xx}(x, y) = -a(ax + by)^{-2}(a) = -\frac{a^2}{(ax + by)^2}, \quad f_{xy}(x, y) = -a(ax + by)^{-2}(b) = -\frac{ab}{(ax + by)^2},$

$f_{yx}(x, y) = -b(ax + by)^{-2}(a) = -\frac{ab}{(ax + by)^2},$ and $f_{yy}(x, y) = -b(ax + by)^{-2}(b) = -\frac{b^2}{(ax + by)^2}.$

49. $z = \frac{y}{2x + 3y} = y(2x + 3y)^{-1} \Rightarrow z_x = y(-1)(2x + 3y)^{-2}(2) = -\frac{2y}{(2x + 3y)^2},$

$z_y = \frac{(2x + 3y) \cdot 1 - y \cdot 3}{(2x + 3y)^2} = \frac{2x}{(2x + 3y)^2}$. Then $z_{xx} = -2y(-2)(2x + 3y)^{-3}(2) = \frac{8y}{(2x + 3y)^3},$

$z_{xy} = -\frac{(2x + 3y)^2 \cdot 2 - 2y \cdot 2(2x + 3y)(3)}{[(2x + 3y)^2]^2} = -\frac{(2x + 3y)(4x + 6y - 12y)}{(2x + 3y)^4} = \frac{6y - 4x}{(2x + 3y)^3},$

$z_{yx} = \frac{(2x + 3y)^2 \cdot 2 - 2x \cdot 2(2x + 3y)(2)}{[(2x + 3y)^2]^2} = \frac{6y - 4x}{(2x + 3y)^3}, \quad z_{yy} = 2x(-2)(2x + 3y)^{-3}(3) = -\frac{12x}{(2x + 3y)^3}.$

50. $T = e^{-2r} \cos \theta \Rightarrow T_r = -2e^{-2r} \cos \theta, \quad T_\theta = -e^{-2r} \sin \theta$. Then $T_{rr} = -2e^{-2r}(-2) \cos \theta = 4e^{-2r} \cos \theta,$

$T_{r\theta} = 2e^{-2r} \sin \theta, \quad T_{\theta r} = -e^{-2r}(-2) \sin \theta = 2e^{-2r} \sin \theta, \quad T_{\theta\theta} = -e^{-2r} \cos \theta.$

51. $v = \sin(s^2 - t^2) \Rightarrow v_s = \cos(s^2 - t^2) \cdot 2s = 2s \cos(s^2 - t^2), \quad v_t = \cos(s^2 - t^2) \cdot (-2t) = -2t \cos(s^2 - t^2)$. Then

$v_{ss} = 2s[-\sin(s^2 - t^2) \cdot 2s] + \cos(s^2 - t^2) \cdot 2 = 2 \cos(s^2 - t^2) - 4s^2 \sin(s^2 - t^2),$

$v_{st} = 2s[-\sin(s^2 - t^2) \cdot (-2t)] + \cos(s^2 - t^2) \cdot 0 = 4st \sin(s^2 - t^2),$

$v_{ts} = -2t[-\sin(s^2 - t^2) \cdot 2s] + \cos(s^2 - t^2) \cdot 0 = 4st \sin(s^2 - t^2),$

$v_{tt} = -2t[-\sin(s^2 - t^2) \cdot (-2t)] + \cos(s^2 - t^2) \cdot (-2) = -2 \cos(s^2 - t^2) - 4t^2 \sin(s^2 - t^2).$

Sec 14.3

$z = -2(x - 1) + 1 = -2x + 3$. So the equation of the tangent line can be given by $z = -2x + 3$, $y = 1$ which has a slope $m = -2$. Therefore, $f_x(1, 1) = -2$.

- (b) In the plane $x = 1$, $f(1, y) = 4 - 1^2 - 2y^2 = 3 - 2y^2$, so a vector equation for C_2 is given by $\mathbf{r}(t) = \langle 1, t, 3 - 2t^2 \rangle$ where the point $(1, 1, 1)$ corresponds to $t = 1$. Then $\mathbf{r}'(t) = \langle 0, 1, -4t \rangle \Rightarrow \mathbf{r}'(1) = \langle 0, 1, -4 \rangle$ and parametric equations of the tangent line are $x = 1, y = t + 1, z = -4t + 1$. Thus, $y = t + 1 \Rightarrow y - 1 = t \Rightarrow z = -4(y - 1) + 1 = -4y + 5$. So the equation of the tangent line can be given by $z = -4y + 5$, $x = 1$ which has a slope $m = -4$. Therefore, $f_y(1, 1) = -4$.

76. For each i , $i = 1, \dots, n$, $\partial u / \partial x_i = a_i e^{a_1 x_1 + a_2 x_2 + \dots + a_n x_n}$ and $\partial^2 u / \partial x_i^2 = a_i^2 e^{a_1 x_1 + a_2 x_2 + \dots + a_n x_n}$.

Then $\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} = (a_1^2 + a_2^2 + \dots + a_n^2) e^{a_1 x_1 + a_2 x_2 + \dots + a_n x_n} = e^{a_1 x_1 + a_2 x_2 + \dots + a_n x_n} = u$

since $a_1^2 + a_2^2 + \dots + a_n^2 = 1$.

77. (a) $u = \sin(kx) \sin(akt) \Rightarrow u_t = ak \sin(kx) \cos(akt)$, $u_{tt} = -a^2 k^2 \sin(kx) \sin(akt)$, $u_x = k \cos(kx) \sin(akt)$, $u_{xx} = -k^2 \sin(kx) \sin(akt)$. Thus $u_{tt} = a^2 u_{xx}$.

(b) $u = \frac{t}{a^2 t^2 - x^2} \Rightarrow u_t = \frac{(a^2 t^2 - x^2) - t(2a^2 t)}{(a^2 t^2 - x^2)^2} = -\frac{a^2 t^2 + x^2}{(a^2 t^2 - x^2)^2}$,

$$u_{tt} = \frac{-2a^2 t(a^2 t^2 - x^2)^2 + (a^2 t^2 + x^2)(2)(a^2 t^2 - x^2)(2a^2 t)}{(a^2 t^2 - x^2)^4} = \frac{2a^4 t^3 + 6a^2 t x^2}{(a^2 t^2 - x^2)^3},$$

$$u_x = t(-1)(a^2 t^2 - x^2)^{-2}(-2x) = \frac{2tx}{(a^2 t^2 - x^2)^2},$$

$$u_{xx} = \frac{2t(a^2 t^2 - x^2)^2 - 2tx(2)(a^2 t^2 - x^2)(-2x)}{(a^2 t^2 - x^2)^4} = \frac{2a^2 t^3 - 2tx^2 + 8tx^2}{(a^2 t^2 - x^2)^3} = \frac{2a^2 t^3 + 6tx^2}{(a^2 t^2 - x^2)^3}.$$

Thus $u_{tt} = a^2 u_{xx}$.

- (c) $u = (x - at)^6 + (x + at)^6 \Rightarrow u_t = -6a(x - at)^5 + 6a(x + at)^5$, $u_{tt} = 30a^2(x - at)^4 + 30a^2(x + at)^4$, $u_x = 6(x - at)^5 + 6(x + at)^5$, $u_{xx} = 30(x - at)^4 + 30(x + at)^4$. Thus $u_{tt} = a^2 u_{xx}$.

- (d) $u = \sin(x - at) + \ln(x + at) \Rightarrow u_t = -a \cos(x - at) + \frac{a}{x + at}$, $u_{tt} = -a^2 \sin(x - at) - \frac{a^2}{(x + at)^2}$, $u_x = \cos(x - at) + \frac{1}{x + at}$, $u_{xx} = -\sin(x - at) - \frac{1}{(x + at)^2}$. Thus $u_{tt} = a^2 u_{xx}$.

78. (a) $u = x^2 + y^2 \Rightarrow u_x = 2x$, $u_{xx} = 2$; $u_y = 2y$, $u_{yy} = 2$. Thus $u_{xx} + u_{yy} \neq 0$ and $u = x^2 + y^2$ does not satisfy Laplace's Equation.

- (b) $u = x^2 - y^2$ is a solution: $u_{xx} = 2$, $u_{yy} = -2$ so $u_{xx} + u_{yy} = 0$.

- (c) $u = x^3 + 3xy^2$ is not a solution: $u_x = 3x^2 + 3y^2$, $u_{xx} = 6x$; $u_y = 6xy$, $u_{yy} = 6x$.

(d) $u = \ln \sqrt{x^2 + y^2}$ is a solution: $u_x = \frac{1}{\sqrt{x^2 + y^2}} \left(\frac{1}{2} \right) (x^2 + y^2)^{-1/2} (2x) = \frac{x}{x^2 + y^2}$,

$$u_{xx} = \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}. \text{ By symmetry, } u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2}, \text{ so } u_{xx} + u_{yy} = 0.$$

(e) $u = \sin x \cosh y + \cos x \sinh y$ is a solution: $u_x = \cos x \cosh y - \sin x \sinh y$, $u_{xx} = -\sin x \cosh y - \cos x \sinh y$,
and $u_y = \sin x \sinh y + \cos x \cosh y$, $u_{yy} = \sin x \cosh y + \cos x \sinh y$.

(f) $u = e^{-x} \cos y - e^{-y} \cos x$ is a solution: $u_x = -e^{-x} \cos y + e^{-y} \sin x$, $u_{xx} = e^{-x} \cos y + e^{-y} \cos x$, and
 $u_y = -e^{-x} \sin y + e^{-y} \cos x$, $u_{yy} = -e^{-x} \cos y - e^{-y} \cos x$.

79. $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \Rightarrow u_x = \left(-\frac{1}{2}\right)(x^2 + y^2 + z^2)^{-3/2}(2x) = -x(x^2 + y^2 + z^2)^{-3/2}$ and

$$u_{xx} = -(x^2 + y^2 + z^2)^{-3/2} - x\left(-\frac{3}{2}\right)(x^2 + y^2 + z^2)^{-5/2}(2x) = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}.$$

By symmetry, $u_{yy} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$ and $u_{zz} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}$.

Thus, $u_{xx} + u_{yy} + u_{zz} = \frac{2x^2 - y^2 - z^2 + 2y^2 - x^2 - z^2 + 2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}} = 0$.

80. $u = e^{-\alpha^2 k^2 t} \sin kx \Rightarrow u_x = k e^{-\alpha^2 k^2 t} \cos kx$, $u_{xx} = -k^2 e^{-\alpha^2 k^2 t} \sin kx$, and $u_t = -\alpha^2 k^2 e^{-\alpha^2 k^2 t} \sin kx$.
Thus, $\alpha^2 u_{xx} = u_t$.

81. $c(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/(4Dt)} \Rightarrow$

$$\begin{aligned} \frac{\partial c}{\partial t} &= \frac{1}{\sqrt{4\pi Dt}} \cdot e^{-x^2/(4Dt)} \left[-x^2(-1)(4Dt)^{-2}(4D) \right] + e^{-x^2/(4Dt)} \cdot \left(-\frac{1}{2}\right) (4\pi Dt)^{-3/2} (4\pi D) \\ &= (4\pi Dt)^{-3/2} \left(4\pi Dt \cdot \frac{x^2}{4Dt^2} - 2\pi D \right) e^{-x^2/(4Dt)} = \frac{2\pi D}{(4\pi Dt)^{3/2}} \left(\frac{x^2}{2Dt} - 1 \right) e^{-x^2/(4Dt)}, \end{aligned}$$

$$\frac{\partial c}{\partial x} = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/(4Dt)} \cdot \frac{-2x}{4Dt} = \frac{-2\pi x}{(4\pi Dt)^{3/2}} e^{-x^2/(4Dt)}, \text{ and}$$

$$\begin{aligned} \frac{\partial^2 c}{\partial x^2} &= \frac{-2\pi}{(4\pi Dt)^{3/2}} \left[x \cdot e^{-x^2/(4Dt)} \cdot \frac{-2x}{4Dt} + e^{-x^2/(4Dt)} \cdot 1 \right] \\ &= \frac{-2\pi}{(4\pi Dt)^{3/2}} \left(-\frac{x^2}{2Dt} + 1 \right) e^{-x^2/(4Dt)} = \frac{2\pi}{(4\pi Dt)^{3/2}} \left(\frac{x^2}{2Dt} - 1 \right) e^{-x^2/(4Dt)}. \end{aligned}$$

Thus, $\frac{\partial c}{\partial t} = \frac{2\pi D}{(4\pi Dt)^{3/2}} \left(\frac{x^2}{2Dt} - 1 \right) e^{-x^2/(4Dt)} = D \left[\frac{2\pi}{(4\pi Dt)^{3/2}} \left(\frac{x^2}{2Dt} - 1 \right) e^{-x^2/(4Dt)} \right] = D \frac{\partial^2 c}{\partial x^2}$.

82. (a) $T(x, y) = \frac{60}{1 + x^2 + y^2} \Rightarrow \frac{\partial T}{\partial x} = -\frac{60(2x)}{(1 + x^2 + y^2)^2}$, so at $(2, 1)$, $T_x = -\frac{240}{(1 + 4 + 1)^2} = -\frac{20}{3}$.