HW # 14 , Sec. 15.2 SOLUTIONS

To find g_{yx}, we first use Fubini's Theorem to find that $\int_a^x \int_c^y f(s,t)\,dt\,ds = \int_c^y \int_a^x f(s,t)\,dt\,ds$, and then use the Fundamental Theorem twice, as above, to get $g_{yx} = f(x,y)$. So $g_{xy} = g_{yx} = f(x,y)$.

15.2 Double Integrals over General Regions

1. $\int_1^5 \int_0^x (8x-2y)\,dy\,dx = \int_1^5 \left[8xy - y^2\right]_{y=0}^{y=x} dx = \int_1^5 [8x(x) - (x)^2 - 8x(0) + (0)^2]dx$

$= \int_1^5 7x^2\,dx = \frac{7}{3}x^3\Big]_1^5 = \frac{7}{3}(125-1) = \frac{868}{3}$

2. $\int_0^2 \int_0^{y^2} x^2y\,dx\,dy = \int_0^2 \left[\frac{1}{3}x^3y\right]_{x=0}^{x=y^2} dy = \int_0^2 \frac{1}{3}y\left[(y^2)^3 - (0)^3\right] dy$

$= \int_0^2 \frac{1}{3}y^7\,dy = \frac{1}{3}\left[\frac{1}{8}y^8\right]_0^2 = \frac{1}{3}(32-0) = \frac{32}{3}$

3. $\int_0^1 \int_0^y xe^{y^3}\,dx\,dy = \int_0^1 \left[\frac{1}{2}x^2e^{y^3}\right]_{x=0}^{x=y} dy = \int_0^1 \frac{1}{2}e^{y^3}\left[(y)^2 - (0)^2\right] dy$

$= \frac{1}{2}\int_0^1 y^2e^{y^3}\,dy = \frac{1}{2}\left[\frac{1}{3}e^{y^3}\right]_0^1 = \frac{1}{2}\cdot\frac{1}{3}\left(e^1 - e^0\right) = \frac{1}{6}(e-1)$

4. $\int_0^{\pi/2} \int_0^x x\sin y\,dy\,dx = \int_0^{\pi/2} [x(-\cos y)]_{y=0}^{y=x}\,dx = \int_0^{\pi/2} (-x\cos x + x)\,dx = \int_0^{\pi/2} (x - x\cos x)\,dx$

$= \left[\frac{1}{2}x^2 - (x\sin x + \cos x)\right]_0^{\pi/2}$ (by integrating by parts in the second term)

$= \left(\frac{1}{2}\cdot\frac{\pi^2}{4} - \frac{\pi}{2} - 0\right) - (0 - 0 - 1) = \frac{\pi^2}{8} - \frac{\pi}{2} + 1$

5. $\int_0^1 \int_0^{s^2} \cos(s^3)\,dt\,ds = \int_0^1 \left[t\cos(s^3)\right]_{t=0}^{t=s^2} ds = \int_0^1 s^2\cos(s^3)\,ds = \frac{1}{3}\sin(s^3)\Big]_0^1 = \frac{1}{3}(\sin 1 - \sin 0) = \frac{1}{3}\sin 1$

6. $\int_0^1 \int_0^{e^v} \sqrt{1+e^v}\,dw\,dv = \int_0^1 \left[w\sqrt{1+e^v}\right]_{w=0}^{w=e^v} dv = \int_0^1 e^v\sqrt{1+e^v}\,dv = \frac{2}{3}(1+e^v)^{3/2}\Big]_0^1$

$= \frac{2}{3}(1+e)^{3/2} - \frac{2}{3}(1+1)^{3/2} = \frac{2}{3}(1+e)^{3/2} - \frac{4}{3}\sqrt{2}$

7. (a) We express the iterated integral as a Type I: $\displaystyle\int_0^2 \int_x^{3x-x^2} 2y\,dy\,dx$. A Type II would require the sum of two integrals.

(b) $\displaystyle\int_0^2 \int_x^{3x-x^2} 2y\,dy\,dx = \int_0^2 \left[y^2\right]_{y=x}^{y=3x-x^2} dx = \int_0^2 \left[(3x-x^2)^2 - (x)^2\right] dx = \int_0^2 \left(8x^2 - 6x^3 + x^4\right) dx$

$= \left[\frac{8}{3}x^3 - \frac{3}{2}x^4 + \frac{1}{5}x^5\right]_0^2 = \frac{64}{3} - 24 + \frac{32}{5} = \frac{56}{15}$

8. (a) We express the iterated integral as a Type II: $\int_0^1 \int_y^{2-y} (x+y)\,dx\,dy$. A Type I would require the sum of two integrals.

(b) $\displaystyle\int_0^1 \int_y^{2-y} (x+y)\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} + xy\right]_{x=y}^{x=2-y} dy = \int_0^1 \left[\left(\frac{(2-y)^2}{2} + (2-y)y\right) - \left(\frac{y^2}{2} + y^2\right)\right] dy$

$= \int_0^1 (2 - 2y^2)\,dy = \left[2y - \frac{2}{3}y^3\right]_0^1 = 2 - \frac{2}{3} = \frac{4}{3}$

9. (a) We express the iterated integral as a Type II. A Type I would require the sum of two integrals. The curves intersect when $\sqrt{x} = x - 2 \;\Rightarrow\; x = x^2 - 4x + 4 \;\Leftrightarrow\; 0 = x^2 - 5x + 4 \;\Leftrightarrow\; (x-4)(x-1) = 0 \;\Leftrightarrow\; x = 1$ or $x = 4$. The point for $x = 1$ is not in D. Thus, the point of intersection of the curves is $(4,2)$ and the integral is $\int_0^2 \int_{y^2}^{y+2} xy\,dx\,dy$.

Sec 15.2

(b) $\int_0^2 \int_{y^2}^{y+2} xy\,dx\,dy = \int_0^2 y\left[\frac{x^2}{2}\right]_{x=y^2}^{x=y+2} dy = \frac{1}{2}\int_0^2 y\left[(y+2)^2 - (y^2)^2\right] dy = \frac{1}{2}\int_0^2 \left[y^3 + 4y^2 + 4y - y^5\right] dy$

$= \frac{1}{2}\left[\frac{1}{4}y^4 + \frac{4}{3}y^3 + 2y^2 - \frac{1}{6}y^6\right]_0^2 = \frac{1}{2}\left(4 + \frac{32}{3} + 8 - \frac{32}{3}\right) = 6$

10. (a) We express the iterated integral as a Type I. A Type II would require the sum of two integrals. The curves intersect when $6 - x = x^2 \;\Leftrightarrow\; x^2 + x - 6 = 0 \;\Leftrightarrow\; (x+3)(x-2) = 0 \;\Leftrightarrow\; x = -3$ or $x = 2$. The point for $x = -3$ is not in D. Thus, the point of intersection of the two curves is $(2, 4)$ and the integral is $\int_0^2 \int_{x^2}^{6-x} x\,dy\,dx$.

(b) $\int_0^2 \int_{x^2}^{6-x} x\,dy\,dx = \int_0^2 x\Big[y\Big]_{y=x^2}^{y=6-x} dx = \int_0^2 x\left[(6-x) - x^2\right] dx = \int_0^2 \left[6x - x^2 - x^3\right] dx$

$= \left[3x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 = 12 - \frac{8}{3} - 4 = \frac{16}{3}$

11. $\iint_D \frac{y}{x^2+1}\,dA = \int_0^4 \int_0^{\sqrt{x}} \frac{y}{x^2+1}\,dy\,dx = \int_0^4 \left[\frac{1}{x^2+1}\cdot\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{x}} dx = \frac{1}{2}\int_0^4 \frac{x}{x^2+1}\,dx$

$= \frac{1}{2}\left[\frac{1}{2}\ln\left|x^2+1\right|\right]_0^4 = \frac{1}{4}\left[\ln\left(x^2+1\right)\right]_0^4 = \frac{1}{4}(\ln 17 - \ln 1) = \frac{1}{4}\ln 17$

12. $\iint_D (2x+y)\,dA = \int_1^2 \int_{y-1}^1 (2x+y)\,dx\,dy = \int_1^2 \left[x^2 + xy\right]_{x=y-1}^{x=1} dy = \int_1^2 \left[1 + y - (y-1)^2 - y(y-1)\right] dy$

$= \int_1^2 (-2y^2 + 4y)\,dy = \left[-\frac{2}{3}y^3 + 2y^2\right]_1^2 = \left(-\frac{16}{3} + 8\right) - \left(-\frac{2}{3} + 2\right) = \frac{4}{3}$

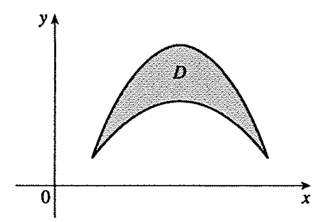
13. $\iint_D e^{-y^2}\,dA = \int_0^3 \int_0^y e^{-y^2}\,dx\,dy = \int_0^3 \left[xe^{-y^2}\right]_{x=0}^{x=y} dy = \int_0^3 \left(ye^{-y^2} - 0\right) dy = \int_0^3 ye^{-y^2}\,dy$

$= -\frac{1}{2}e^{-y^2}\Big]_0^3 = -\frac{1}{2}\left(e^{-9} - e^0\right) = \frac{1}{2}\left(1 - e^{-9}\right)$

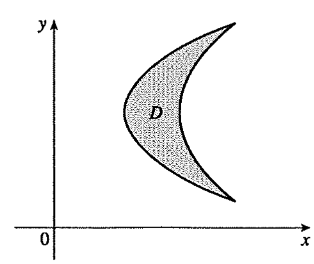
14. $\iint_D y\sqrt{x^2 - y^2}\,dA = \int_0^2 \int_0^x y\sqrt{x^2 - y^2}\,dy\,dx = \int_0^2 \left[-\frac{1}{3}(x^2 - y^2)^{3/2}\right]_{y=0}^{y=x} dx = \int_0^2 \left[0 + \frac{1}{3}(x^2)^{3/2}\right] dx$

$= \int_0^2 \frac{1}{3}x^3\,dx = \frac{1}{3}\cdot\frac{1}{4}x^4\Big]_0^2 = \frac{1}{12}(16 - 0) = \frac{4}{3}$

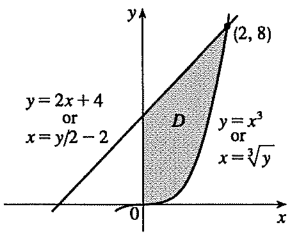
15. (a) At the right we sketch an example of a region D that can be described as lying between the graphs of two continuous functions of x (a type I region) but not as lying between graphs of two continuous functions of y (a type II region). The regions shown in Figures 6 and 8 in the text are additional examples.



(b) Now we sketch an example of a region D that can be described as lying between the graphs of two continuous functions of y but not as lying between graphs of two continuous functions of x. The first region shown in Figure 7 is another example.



22.



By inspection, the curves $y = 2x + 4$ and $y = x^3$ intersect when $x^3 = 2x + 4 \Leftrightarrow x = 2$, so the point of intersection is $(2, 8)$. If we describe D as a type 1 region, $D = \{(x, y) \mid 0 \le x \le 2,\, x^3 \le y \le 2x + 4\}$ and the integral is

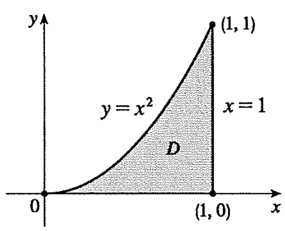
$$\iint_D 6x^2\,dA = \int_0^2 \int_{x^3}^{2x+4} 6x^2\,dy\,dx.$$

If we describe D as a type II region, the right boundary curve is $x = \sqrt[3]{y}$, but the left boundary curve consists of two parts, $x = 0$ for $0 \le y \le 4$ and $x = y/2 - 2$ for $4 \le y \le 8$.

In either case, the resulting iterated integrals are not difficult to evaluate, but the region D is more simply described as a type I region, giving one iterated integral rather than a sum of two, so we evaluate that integral:

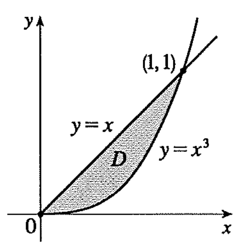
$$\int_0^2 \int_{x^3}^{2x+4} 6x^2\,dy\,dx = \int_0^2 \Big[6x^2 y\Big]_{y=x^3}^{y=2x+4} dx = \int_0^2 [6x^2(2x + 4 - x^3)]\,dx = \int_0^2 (12x^3 + 24x^2 - 6x^5)\,dx$$
$$= \big[3x^4 + 8x^3 - x^6\big]_0^2 = 48 + 64 - 64 = 48$$

23.



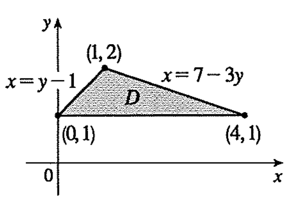
$$\int_0^1 \int_0^{x^2} x\cos y\,dy\,dx = \int_0^1 \big[x\sin y\big]_{y=0}^{y=x^2} dx = \int_0^1 x\sin x^2\,dx$$
$$= -\tfrac{1}{2}\cos x^2\big]_0^1 = -\tfrac{1}{2}(\cos 1 - \cos 0) = \tfrac{1}{2}(1 - \cos 1)$$

24.



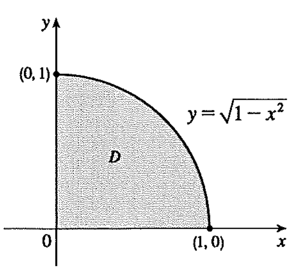
$$\iint_D (x^2 + 2y)\,dA = \int_0^1 \int_{x^3}^{x} (x^2 + 2y)\,dy\,dx = \int_0^1 \big[x^2 y + y^2\big]_{y=x^3}^{y=x} dx$$
$$= \int_0^1 (x^3 + x^2 - x^5 - x^6)\,dx = \big[\tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 - \tfrac{1}{6}x^6 - \tfrac{1}{7}x^7\big]_0^1$$
$$= \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{7} = \tfrac{23}{84}$$

25.



$$\iint_D y^2\,dA = \int_1^2 \int_{y-1}^{7-3y} y^2\,dx\,dy = \int_1^2 \big[xy^2\big]_{x=y-1}^{x=7-3y} dy$$
$$= \int_1^2 [(7 - 3y) - (y - 1)]\,y^2\,dy = \int_1^2 (8y^2 - 4y^3)\,dy$$
$$= \big[\tfrac{8}{3}y^3 - y^4\big]_1^2 = \tfrac{64}{3} - 16 - \tfrac{8}{3} + 1 = \tfrac{11}{3}$$

26.



$$\iint_D xy\,dA = \int_0^1 \int_0^{\sqrt{1-x^2}} xy\,dy\,dx$$
$$= \int_0^1 \big[\tfrac{1}{2}xy^2\big]_{y=0}^{y=\sqrt{1-x^2}} dx = \int_0^1 \tfrac{1}{2}x(1 - x^2)\,dx$$
$$= \tfrac{1}{2}\int_0^1 (x - x^3)\,dx = \tfrac{1}{2}\big[\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\big]_0^1$$
$$= \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{4} - 0\right) = \tfrac{1}{8}$$

51. The two bounding curves $y = x^3 - x$ and $y = x^2 + x$ intersect at the origin and at $x = 2$, with $x^2 + x > x^3 - x$ on $(0, 2)$. Using a CAS, we find that the volume of the solid is

$$V = \int_0^2 \int_{x^3 - x}^{x^2 + x} (x^3 y^4 + xy^2)\, dy\, dx = \frac{13{,}984{,}735{,}616}{14{,}549{,}535}$$

52. For $|x| \le 1$ and $|y| \le 1$, $2x^2 + y^2 < 8 - x^2 - 2y^2$. Also, the cylinder is described by the inequalities $-1 \le x \le 1$, $-\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}$. So the volume is given by

$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [(8 - x^2 - 2y^2) - (2x^2 + y^2)]\, dy\, dx = \frac{13\pi}{2} \qquad \text{[using a CAS]}$$

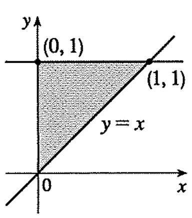
53. The two surfaces intersect in the circle $x^2 + y^2 = 1$, $z = 0$ and the region of integration is the disk D: $x^2 + y^2 \le 1$.

Using a CAS, the volume is $\displaystyle\iint_D (1 - x^2 - y^2)\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx = \frac{\pi}{2}$.

54. The projection onto the xy-plane of the intersection of the two surfaces is the circle $x^2 + y^2 = 2y \;\Rightarrow$ $x^2 + y^2 - 2y = 0 \;\Rightarrow\; x^2 + (y - 1)^2 = 1$, so the region of integration is given by $-1 \le x \le 1$, $1 - \sqrt{1 - x^2} \le y \le 1 + \sqrt{1 - x^2}$. In this region, $2y \ge x^2 + y^2$ so, using a CAS, the volume is

$$V = \int_{-1}^{1} \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} [2y - (x^2 + y^2)]\, dy\, dx = \frac{\pi}{2}$$

55.

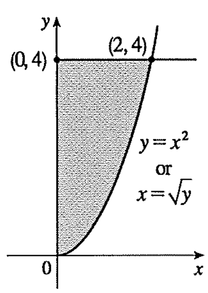


Because the region of integration is

$D = \{(x, y) \mid 0 \le x \le y, 0 \le y \le 1\} = \{(x, y) \mid x \le y \le 1, 0 \le x \le 1\}$

we have $\int_0^1 \int_0^y f(x, y)\, dx\, dy = \iint_D f(x, y)\, dA = \int_0^1 \int_x^1 f(x, y)\, dy\, dx$.

56.

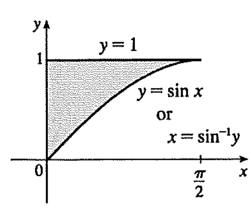


Because the region of integration is

$$D = \{(x, y) \mid x^2 \le y \le 4, 0 \le x \le 2\}$$
$$= \{(x, y) \mid 0 \le x \le \sqrt{y}, 0 \le y \le 4\}$$

we have $\int_0^2 \int_{x^2}^4 f(x, y)\, dy\, dx = \iint_D f(x, y)\, dA = \int_0^4 \int_0^{\sqrt{y}} f(x, y)\, dx\, dy$.

57.



Because the region of integration is

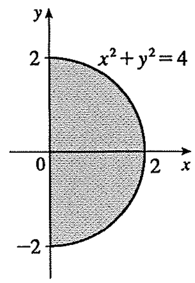
$$D = \{(x, y) \mid 0 \le x \le \pi/2,\ \sin x \le y \le 1\}$$
$$= \{(x, y) \mid 0 \le x \le \sin^{-1} y, 0 \le y \le 1\}$$

we have

$\int_0^{\pi/2} \int_{\sin x}^1 f(x, y)\, dy\, dx = \iint_D f(x, y)\, dA = \int_0^1 \int_0^{\sin^{-1} y} f(x, y)\, dx\, dy$

Sec 15.2

58.



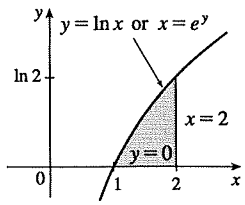
Because the region of integration is

$$D=\left\{(x,y)\mid 0\le x\le\sqrt{4-y^2},\,-2\le y\le 2\right\}$$
$$=\left\{(x,y)\mid -\sqrt{4-x^2}\le y\le\sqrt{4-x^2},\,0\le x\le 2\right\}$$

we have

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}f(x,y)\,dx\,dy=\iint_D f(x,y)\,dA=\int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}f(x,y)\,dy\,dx.$$

59.



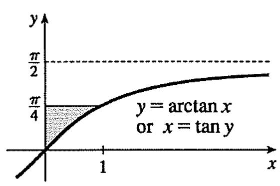
Because the region of integration is

$$D=\{(x,y)\mid 0\le y\le\ln x,\,1\le x\le 2\}=\{(x,y)\mid e^y\le x\le 2,\,0\le y\le\ln 2\}$$

we have

$$\int_1^2\int_0^{\ln x}f(x,y)\,dy\,dx=\iint_D f(x,y)\,dA=\int_0^{\ln 2}\int_{e^y}^{2}f(x,y)\,dx\,dy$$

60.



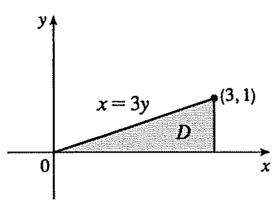
Because the region of integration is

$$D=\left\{(x,y)\mid \arctan x\le y\le\tfrac{\pi}{4},\,0\le x\le 1\right\}$$
$$=\left\{(x,y)\mid 0\le x\le\tan y,\,0\le y\le\tfrac{\pi}{4}\right\}$$

we have

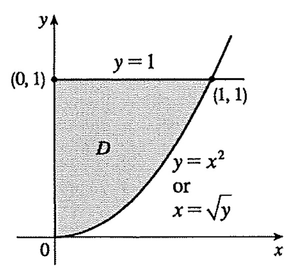
$$\int_0^1\int_{\arctan x}^{\pi/4}f(x,y)\,dy\,dx=\iint_D f(x,y)\,dA=\int_0^{\pi/4}\int_0^{\tan y}f(x,y)\,dx\,dy$$

61.



$$\int_0^1\int_{3y}^{3}e^{x^2}\,dx\,dy=\int_0^3\int_0^{x/3}e^{x^2}\,dy\,dx=\int_0^3\left[e^{x^2}y\right]_{y=0}^{y=x/3}dx$$
$$=\int_0^3\left(\frac{x}{3}\right)e^{x^2}\,dx=\tfrac{1}{6}\,e^{x^2}\Big]_0^3=\frac{e^9-1}{6}$$

62.

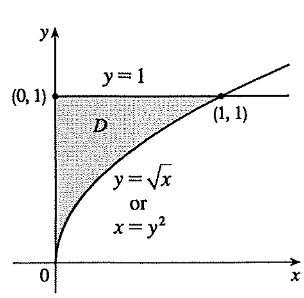


$$\int_0^1\int_{x^2}^{1}\sqrt{y}\,\sin y\,dy\,dx=\int_0^1\int_0^{\sqrt{y}}\sqrt{y}\,\sin y\,dx\,dy=\int_0^1\sqrt{y}\,\sin y\,[x]_{x=0}^{x=\sqrt{y}}\,dy$$
$$=\int_0^1\left(\sqrt{y}\,\sin y\right)\left(\sqrt{y}-0\right)dy=\int_0^1 y\,\sin y\,dy$$
$$=-y\cos y\big]_0^1+\int_0^1\cos y\,dy$$

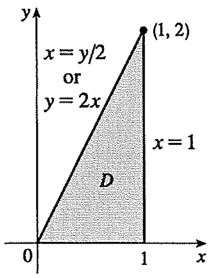
[by integrating by parts with $u=y$, $dv=\sin y\,dy$]

$$=\left[-y\cos y+\sin y\right]_0^1=-\cos 1+\sin 1-0=\sin 1-\cos 1$$

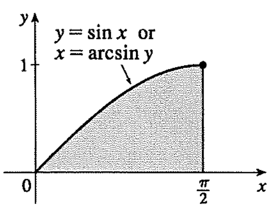
63.



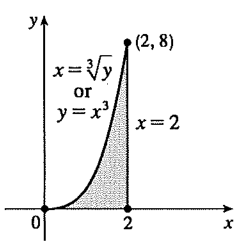
$$\int_0^1\int_{\sqrt{x}}^{1}\sqrt{y^3+1}\,dy\,dx=\int_0^1\int_0^{y^2}\sqrt{y^3+1}\,dx\,dy=\int_0^1\sqrt{y^3+1}\,[x]_{x=0}^{x=y^2}\,dy$$
$$=\int_0^1 y^2\sqrt{y^3+1}\,dy=\tfrac{2}{9}\left(y^3+1\right)^{3/2}\Big]_0^1$$
$$=\tfrac{2}{9}\left(2^{3/2}-1^{3/2}\right)=\tfrac{2}{9}\left(2\sqrt{2}-1\right)$$

64. 

$$\int_0^2 \int_{y/2}^1 y\cos(x^3-1)\,dx\,dy = \int_0^1 \int_0^{2x} y\cos(x^3-1)\,dy\,dx$$
$$= \int_0^1 \cos(x^3-1)\,\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=2x}\,dx$$
$$= \int_0^1 2x^2\cos(x^3-1)\,dx = \tfrac{2}{3}\sin(x^3-1)\big]_0^1$$
$$= \tfrac{2}{3}\left[0 - \sin(-1)\right] = -\tfrac{2}{3}\sin(-1) = \tfrac{2}{3}\sin 1$$

65. 

$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,dx\,dy$$
$$= \int_0^{\pi/2} \int_0^{\sin x} \cos x\,\sqrt{1+\cos^2 x}\,dy\,dx$$
$$= \int_0^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,\left[y\right]_{y=0}^{y=\sin x}\,dx$$
$$= \int_0^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,\sin x\,dx \qquad \left[\begin{array}{c}\text{Let } u = \cos x,\ du = -\sin x\,dx,\\ dx = du/(-\sin x)\end{array}\right]$$
$$= \int_1^0 -u\sqrt{1+u^2}\,du = -\tfrac{1}{3}\left(1+u^2\right)^{3/2}\Big]_1^0$$
$$= \tfrac{1}{3}\left(\sqrt{8}-1\right) = \tfrac{1}{3}\left(2\sqrt{2}-1\right)$$

66. 

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy = \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx$$
$$= \int_0^2 e^{x^4}\left[y\right]_{y=0}^{y=x^3}\,dx = \int_0^2 x^3 e^{x^4}\,dx$$
$$= \tfrac{1}{4}e^{x^4}\Big]_0^2 = \tfrac{1}{4}(e^{16}-1)$$

67. $D = \{(x,y) \mid 0 \le x \le 1,\ -x+1 \le y \le 1\} \cup \{(x,y) \mid -1 \le x \le 0, x+1 \le y \le 1\}$
$\cup \{(x,y) \mid 0 \le x \le 1,\ -1 \le y \le x-1\} \cup \{(x,y) \mid -1 \le x \le 0,\ -1 \le y \le -x-1\}$, all type I.

$$\iint_D x^2\,dA = \int_0^1 \int_{1-x}^1 x^2\,dy\,dx + \int_{-1}^0 \int_{x+1}^1 x^2\,dy\,dx + \int_0^1 \int_{-1}^{x-1} x^2\,dy\,dx + \int_{-1}^0 \int_{-1}^{-x-1} x^2\,dy\,dx$$
$$= 4\int_0^1 \int_{1-x}^1 x^2\,dy\,dx \qquad \left[\text{by symmetry of the regions and because } f(x,y) = x^2 \ge 0\right]$$
$$= 4\int_0^1 x^3\,dx = 4\left[\tfrac{1}{4}x^4\right]_0^1 = 1$$

68. $D = \left\{(x,y) \mid -1 \le y \le 0,\ -1 \le x \le y - y^3\right\} \cup \left\{(x,y) \mid 0 \le y \le 1,\ \sqrt{y} - 1 \le x \le y - y^3\right\}$, both type II.

$$\iint_D y\,dA = \int_{-1}^0 \int_{-1}^{y-y^3} y\,dx\,dy + \int_0^1 \int_{\sqrt{y}-1}^{y-y^3} y\,dx\,dy = \int_{-1}^0 \left[xy\right]_{x=-1}^{x=y-y^3} dy + \int_0^1 \left[xy\right]_{x=\sqrt{y}-1}^{x=y-y^3} dy$$
$$= \int_{-1}^0 (y^2 - y^4 + y)\,dy + \int_0^1 (y^2 - y^4 - y^{3/2} + y)\,dy$$
$$= \left[\tfrac{1}{3}y^3 - \tfrac{1}{5}y^5 + \tfrac{1}{2}y^2\right]_{-1}^0 + \left[\tfrac{1}{3}y^3 - \tfrac{1}{5}y^5 - \tfrac{2}{5}y^{5/2} + \tfrac{1}{2}y^2\right]_0^1$$
$$= \left(0 - \tfrac{11}{30}\right) + \left(\tfrac{7}{30} - 0\right) = -\tfrac{2}{15}$$