

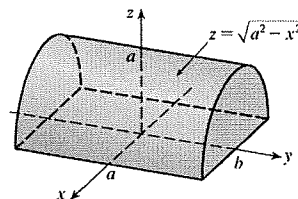
HW #15, Section 15.3 Solutions

1516 □ CHAPTER 15 MULTIPLE INTEGRALS

$\iint_D \sqrt{a^2 - x^2} dA$ represents the volume of the solid region under the graph of $z = \sqrt{a^2 - x^2}$ and above the rectangle D , namely a half circular cylinder with radius a and length $2b$ (see the figure) whose volume is

$$\frac{1}{2} \cdot \pi r^2 h = \frac{1}{2} \pi a^2 (2b) = \pi a^2 b. \text{ Thus}$$

$$\iint_D (ax^3 + by^3 + \sqrt{a^2 - x^2}) dA = 0 + 0 + \pi a^2 b = \pi a^2 b.$$



80. By the Extreme Value Theorem (14.7.8), f has an absolute minimum value m and an absolute maximum value M in D . Then by Property 15.2.10, $m A(D) \leq \iint_D f(x, y) dA \leq M A(D)$. Dividing through by the positive number $A(D)$, we get

$m \leq \frac{1}{A(D)} \iint_D f(x, y) dA \leq M$. This says that the average value of f over D lies between m and M . But f is continuous on D and takes on the values m and M , and so by the Intermediate Value Theorem must take on all values between m and M .

Specifically, there exists a point (x_0, y_0) in D such that $f(x_0, y_0) = \frac{1}{A(D)} \iint_D f(x, y) dA$ or equivalently

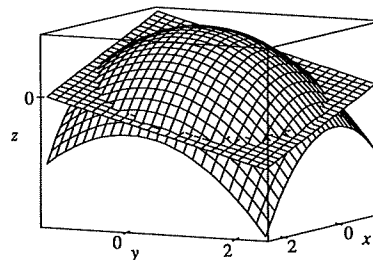
$$\iint_D f(x, y) dA = f(x_0, y_0) A(D).$$

81. For each r such that D_r lies within the domain, $A(D_r) = \pi r^2$, and by the Mean Value Theorem for double integrals there

exists (x_r, y_r) in D_r such that $f(x_r, y_r) = \frac{1}{\pi r^2} \iint_{D_r} f(x, y) dA$. But $\lim_{r \rightarrow 0^+} (x_r, y_r) = (a, b)$,

so $\lim_{r \rightarrow 0^+} \frac{1}{\pi r^2} \iint_{D_r} f(x, y) dA = \lim_{r \rightarrow 0^+} f(x_r, y_r) = f(a, b)$ by the continuity of f .

82. To find the equations of the boundary curves, we require that the z -values of the two surfaces be the same. In Maple, we use the command `solve(4-x^2-y^2=1-x-y, y)`; and in Mathematica, we use `Solve[4-x^2-y^2==1-x-y, y]`. We find that the curves have equations $y = \frac{1 \pm \sqrt{13 + 4x - 4x^2}}{2}$. To find the two points of intersection of these curves, we use the CAS to solve $13 + 4x - 4x^2 = 0$, finding that $x = \frac{1 \pm \sqrt{14}}{2}$. So, using the CAS to evaluate the integral, the volume of intersection is



$$V = \int_{(1-\sqrt{14})/2}^{(1+\sqrt{14})/2} \int_{(1-\sqrt{13+4x-4x^2})/2}^{(1+\sqrt{13+4x-4x^2})/2} [(4-x^2-y^2) - (1-x-y)] dy dx = \frac{49\pi}{8}$$

15.3 Double Integrals in Polar Coordinates

1. The region R is more easily described with polar coordinates: $R = \{(r, \theta) \mid 0 \leq r \leq 4, 0 \leq \theta \leq 3\pi/2\}$.

$$\text{Thus, } \iint_R f(x, y) dA = \int_0^{3\pi/2} \int_0^4 f(r \cos \theta, r \sin \theta) r dr d\theta.$$

2. The region R is more easily described by rectangular coordinates: $R = \{(x, y) \mid -1 \leq x \leq 1, -x \leq y \leq 1\}$.

$$\text{Thus, } \iint_R f(x, y) dA = \int_{-1}^1 \int_{-x}^1 f(x, y) dy dx.$$

3. The region R is more easily described with polar coordinates: $R = \{(r, \theta) \mid 1 \leq r \leq 3, 0 \leq \theta \leq \pi\}$.

$$\text{Thus, } \iint_R f(x, y) dA = \int_0^\pi \int_1^3 f(r \cos \theta, r \sin \theta) r dr d\theta.$$

4. The region R is more easily described by polar coordinates: $R = \{(r, \theta) \mid 0 \leq r \leq 3, -\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}\}$.

$$\text{Thus, } \iint_R f(x, y) dA = \int_{-\pi/4}^{3\pi/4} \int_0^3 f(r \cos \theta, r \sin \theta) r dr d\theta.$$

5. The region R is more easily described with rectangular coordinates: $R = \{(x, y) \mid 2y - 2 \leq x \leq -2y + 2, 0 \leq y \leq 1\}$.

$$\text{Thus, } \iint_R f(x, y) dA = \int_0^1 \int_{2y-2}^{-2y+2} f(x, y) dx dy.$$

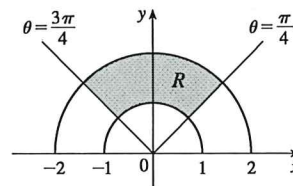
6. The region R is more easily described with polar coordinates: $R = \{(r, \theta) \mid 8 \leq r \leq 10, 0 \leq \theta \leq 2\pi\}$.

$$\text{Thus, } \iint_R f(x, y) dA = \int_0^{2\pi} \int_8^{10} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

7. The integral $\int_{\pi/4}^{3\pi/4} \int_1^2 r dr d\theta$ represents the area of the region

$R = \{(r, \theta) \mid 1 \leq r \leq 2, \pi/4 \leq \theta \leq 3\pi/4\}$, the top quarter portion of a ring (annulus).

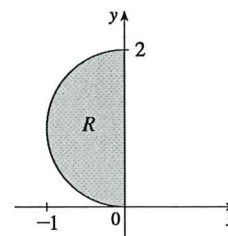
$$\begin{aligned} \int_{\pi/4}^{3\pi/4} \int_1^2 r dr d\theta &= \left(\int_{\pi/4}^{3\pi/4} d\theta \right) \left(\int_1^2 r dr \right) \\ &= [\theta]_{\pi/4}^{3\pi/4} \left[\frac{1}{2} r^2 \right]_1^2 = \left(\frac{3\pi}{4} - \frac{\pi}{4} \right) \cdot \frac{1}{2} (4 - 1) = \frac{\pi}{2} \cdot \frac{3}{2} = \frac{3\pi}{4} \end{aligned}$$



8. The integral $\int_{\pi/2}^\pi \int_0^{2\sin\theta} r dr d\theta$ represents the area of the region $R = \{(r, \theta) \mid 0 \leq r \leq 2\sin\theta, \pi/2 \leq \theta \leq \pi\}$. Since

$r = 2\sin\theta \Rightarrow r^2 = 2r\sin\theta \Leftrightarrow x^2 + y^2 = 2y \Leftrightarrow x^2 + (y-1)^2 = 1$, R is the portion in the second quadrant of a disk of radius 1 with center $(0, 1)$.

$$\begin{aligned} \int_{\pi/2}^\pi \int_0^{2\sin\theta} r dr d\theta &= \int_{\pi/2}^\pi \left[\frac{1}{2} r^2 \right]_{r=0}^{r=2\sin\theta} d\theta = \int_{\pi/2}^\pi 2\sin^2\theta d\theta \\ &= \int_{\pi/2}^\pi 2 \cdot \frac{1}{2} (1 - \cos 2\theta) d\theta = [\theta - \frac{1}{2} \sin 2\theta]_{\pi/2}^\pi \\ &= \pi - 0 - \frac{\pi}{2} + 0 = \frac{\pi}{2} \end{aligned}$$

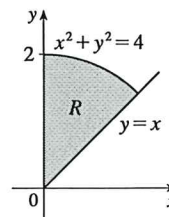


9. The half-disk D can be described in polar coordinates as $D = \{(r, \theta) \mid 0 \leq r \leq 5, 0 \leq \theta \leq \pi\}$. Then

$$\begin{aligned} \iint_D x^2 y dA &= \int_0^\pi \int_0^5 (r \cos \theta)^2 (r \sin \theta) r dr d\theta = \left(\int_0^\pi \cos^2 \theta \sin \theta d\theta \right) \left(\int_0^5 r^4 dr \right) \\ &= \left[-\frac{1}{3} \cos^3 \theta \right]_0^\pi \left[\frac{1}{5} r^5 \right]_0^5 = -\frac{1}{3} (-1 - 1) \cdot 625 = \frac{1250}{3} \end{aligned}$$

10. The region R is $\frac{1}{8}$ of a disk, as shown in the figure, and can be described by $R = \{(r, \theta) \mid 0 \leq r \leq 2, \pi/4 \leq \theta \leq \pi/2\}$. Thus

$$\begin{aligned} \iint_R (2x - y) dA &= \int_{\pi/4}^{\pi/2} \int_0^2 (2r \cos \theta - r \sin \theta) r dr d\theta \\ &= \int_{\pi/4}^{\pi/2} (2 \cos \theta - \sin \theta) d\theta \int_0^2 r^2 dr \\ &= [2 \sin \theta + \cos \theta]_{\pi/4}^{\pi/2} \left[\frac{1}{3} r^3 \right]_0^2 \\ &= (2 + 0 - \sqrt{2} - \frac{\sqrt{2}}{2}) \left(\frac{8}{3} \right) = \frac{16}{3} - 4\sqrt{2} \end{aligned}$$



$$\begin{aligned} 11. \iint_R \sin(x^2 + y^2) dA &= \int_0^{\pi/2} \int_1^3 \sin(r^2) r dr d\theta = \int_0^{\pi/2} d\theta \int_1^3 r \sin(r^2) dr = [\theta]_0^{\pi/2} \left[-\frac{1}{2} \cos(r^2)\right]_1^3 \\ &= \left(\frac{\pi}{2}\right) \left[-\frac{1}{2}(\cos 9 - \cos 1)\right] = \frac{\pi}{4}(\cos 1 - \cos 9) \end{aligned}$$

$$\begin{aligned} 12. \iint_R \frac{y^2}{x^2 + y^2} dA &= \int_0^{2\pi} \int_a^b \frac{(r \sin \theta)^2}{r^2} r dr d\theta = \int_0^{2\pi} \sin^2 \theta d\theta \int_a^b r dr = \int_0^{2\pi} \frac{1}{2}(1 - \cos 2\theta) d\theta \int_a^b r dr \\ &= \frac{1}{2} \left[\theta - \frac{1}{2} \sin 2\theta\right]_0^{2\pi} \left[\frac{1}{2} r^2\right]_a^b = \frac{1}{2} (2\pi - 0 - 0) \cdot \frac{1}{2} (b^2 - a^2) = \frac{\pi}{2} (b^2 - a^2) \end{aligned}$$

$$\begin{aligned} 13. \iint_D e^{-x^2-y^2} dA &= \int_{-\pi/2}^{\pi/2} \int_0^2 e^{-r^2} r dr d\theta = \int_{-\pi/2}^{\pi/2} d\theta \int_0^2 r e^{-r^2} dr \\ &= [\theta]_{-\pi/2}^{\pi/2} \left[-\frac{1}{2} e^{-r^2}\right]_0^2 = \pi \left(-\frac{1}{2}\right)(e^{-4} - e^0) = \frac{\pi}{2}(1 - e^{-4}) \end{aligned}$$

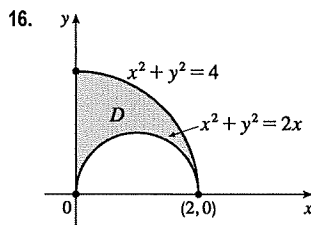
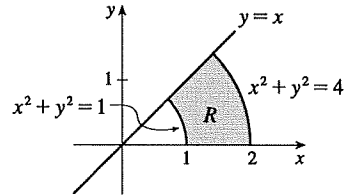
$$\begin{aligned} 14. \iint_D \cos \sqrt{x^2 + y^2} dA &= \int_0^{2\pi} \int_0^2 \cos \sqrt{r^2} r dr d\theta = \int_0^{2\pi} d\theta \int_0^2 r \cos r dr. \text{ For the second integral, integrate by parts with } \\ u &= r, dv = \cos r dr. \text{ Then } \iint_D \cos \sqrt{x^2 + y^2} dA = [\theta]_0^{2\pi} [r \sin r + \cos r]_0^2 = 2\pi(2 \sin 2 + \cos 2 - 1). \end{aligned}$$

15. R is the region shown in the figure, and can be described by $R = \{(r, \theta) \mid 0 \leq \theta \leq \pi/4, 1 \leq r \leq 2\}$. Thus

$$\iint_R \arctan(y/x) dA = \int_0^{\pi/4} \int_1^2 \arctan(\tan \theta) r dr d\theta \text{ since } y/x = \tan \theta.$$

Also, $\arctan(\tan \theta) = \theta$ for $0 \leq \theta \leq \pi/4$, so the integral becomes

$$\int_0^{\pi/4} \int_1^2 \theta r dr d\theta = \int_0^{\pi/4} \theta d\theta \int_1^2 r dr = \left[\frac{1}{2} \theta^2\right]_0^{\pi/4} \left[\frac{1}{2} r^2\right]_1^2 = \frac{\pi^2}{32} \cdot \frac{3}{2} = \frac{3}{64} \pi^2.$$

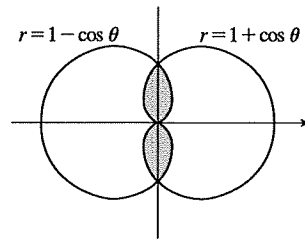


$$\begin{aligned} \iint_D x dA &= \iint_{\substack{x^2+y^2 \leq 4 \\ x \geq 0, y \geq 0}} x dA - \iint_{\substack{(x-1)^2+y^2 \leq 1 \\ y \geq 0}} x dA \\ &= \int_0^{\pi/2} \int_0^2 r^2 \cos \theta dr d\theta - \int_0^{\pi/2} \int_0^{2 \cos \theta} r^2 \cos \theta dr d\theta \\ &= \int_0^{\pi/2} \frac{1}{3} (8 \cos \theta) d\theta - \int_0^{\pi/2} \frac{1}{3} (8 \cos^4 \theta) d\theta \\ &= \frac{8}{3} [\sin \theta]_0^{\pi/2} - \frac{8}{12} [\cos^3 \theta \sin \theta + \frac{3}{2} (\theta + \sin \theta \cos \theta)]_0^{\pi/2} \\ &= \frac{8}{3} - \frac{2}{3} \left[0 + \frac{3}{2} \left(\frac{\pi}{2}\right)\right] = \frac{16-3\pi}{6} \end{aligned}$$

17. By symmetry, the area of the region is 4 times the area of the region D in the first quadrant enclosed by the cardioid

$r = 1 - \cos \theta$ (see the figure). Here $D = \{(r, \theta) \mid 0 \leq r \leq 1 - \cos \theta, 0 \leq \theta \leq \pi/2\}$, so the total area is

$$\begin{aligned} 4A(D) &= 4 \iint_D dA = 4 \int_0^{\pi/2} \int_0^{1-\cos \theta} r dr d\theta = 4 \int_0^{\pi/2} \left[\frac{1}{2} r^2\right]_{r=0}^{r=1-\cos \theta} d\theta \\ &= 2 \int_0^{\pi/2} (1 - \cos \theta)^2 d\theta = 2 \int_0^{\pi/2} (1 - 2 \cos \theta + \cos^2 \theta) d\theta \\ &= 2 \int_0^{\pi/2} \left[1 - 2 \cos \theta + \frac{1}{2} (1 + \cos 2\theta)\right] d\theta \\ &= 2 \left[\theta - 2 \sin \theta + \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta\right]_0^{\pi/2} \\ &= 2 \left(\frac{\pi}{2} - 2 + \frac{\pi}{4}\right) = \frac{3\pi}{2} - 4 \end{aligned}$$



18. The region D is described by $D = \{(r, \theta) \mid 0 \leq r \leq \sqrt{\theta}, 0 \leq \theta \leq 2\pi\}$, so the area is

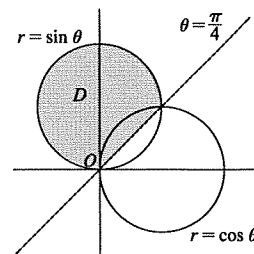
$$A(D) = \int_0^{2\pi} \int_0^{\sqrt{\theta}} r \, dr \, d\theta = \int_0^{2\pi} \left[\frac{r^2}{2} \right]_{r=0}^{r=\sqrt{\theta}} d\theta = \int_0^{2\pi} \frac{\theta}{2} d\theta = \left[\frac{\theta^2}{4} \right]_0^{2\pi} = \pi^2.$$

19. By symmetry, the total area is twice the area defined by

$$D = \{(r, \theta) \mid 0 \leq r \leq \sin \theta, \pi/4 \leq \theta \leq \pi\} \text{ (see the figure).}$$

The total area is

$$\begin{aligned} 2A(D) &= 2 \int_{\pi/4}^{\pi} \int_0^{\sin \theta} r \, dr \, d\theta = 2 \cdot \frac{1}{2} \int_{\pi/4}^{\pi} [r^2]_{r=0}^{r=\sin \theta} d\theta = \int_{\pi/4}^{\pi} \sin^2 \theta \, d\theta \\ &= \int_{\pi/4}^{\pi} \frac{1}{2} (1 - \cos 2\theta) \, d\theta = \frac{1}{2} \left[\theta - \frac{1}{2} \sin 2\theta \right]_{\pi/4}^{\pi} \\ &= \frac{1}{2} (\pi - 0) - \frac{1}{2} \left(\frac{\pi}{4} - \frac{1}{2} \right) = \frac{3\pi}{8} + \frac{1}{4} \end{aligned}$$



20. By symmetry, the area of the region is 4 times the area of the region D in the first quadrant between the circle $r = 1/\sqrt{2}$ and

the curve $r^2 = \cos 2\theta \Rightarrow r = \sqrt{\cos 2\theta}$. The curves intersect in the first quadrant when $\cos 2\theta = \left(\frac{1}{\sqrt{2}}\right)^2 \Rightarrow$

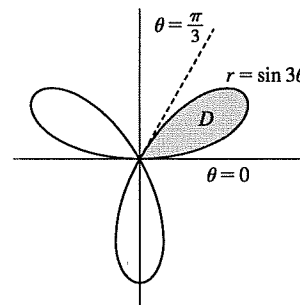
$\cos 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{6}$. Thus, $D = \{(r, \theta) \mid 1/\sqrt{2} \leq r \leq \sqrt{\cos 2\theta}, 0 \leq \theta \leq \pi/6\}$, so the total area is

$$\begin{aligned} 4A(D) &= 4 \int_0^{\pi/6} \int_{1/\sqrt{2}}^{\sqrt{\cos 2\theta}} r \, dr \, d\theta = 4 \cdot \frac{1}{2} \int_0^{\pi/6} [r^2]_{r=1/\sqrt{2}}^{r=\sqrt{\cos 2\theta}} d\theta = 2 \int_0^{\pi/6} \left[\cos 2\theta - \frac{1}{2} \right] d\theta \\ &= 2 \left[\frac{1}{2} \sin 2\theta - \frac{\theta}{2} \right]_0^{\pi/6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6} \end{aligned}$$

21. One loop is given by the region

$$D = \{(r, \theta) \mid 0 \leq r \leq \sin 3\theta, 0 \leq \theta \leq \pi/3\}, \text{ so the area is}$$

$$\begin{aligned} \iint_D dA &= \int_0^{\pi/3} \int_0^{\sin 3\theta} r \, dr \, d\theta = \frac{1}{2} \int_0^{\pi/3} [r^2]_{r=0}^{r=\sin 3\theta} d\theta \\ &= \frac{1}{2} \int_0^{\pi/3} \sin^2 3\theta \, d\theta = \frac{1}{2} \int_0^{\pi/3} \frac{1}{2} (1 - \cos 6\theta) \, d\theta \\ &= \frac{1}{4} \left[\theta - \frac{1}{6} \sin 6\theta \right]_0^{\pi/3} = \frac{\pi}{12} \end{aligned}$$



22. In polar coordinates the circle $(x-1)^2 + y^2 = 1 \Leftrightarrow x^2 + y^2 = 2x$ is $r^2 = 2r \cos \theta \Rightarrow r = 2 \cos \theta$,

and the circle $x^2 + y^2 = 1$ is $r = 1$. The curves intersect in the first quadrant when

$2 \cos \theta = 1 \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \pi/3$, so the portion of the region in the first quadrant is given by

$D = \{(r, \theta) \mid 1 \leq r \leq 2 \cos \theta, 0 \leq \theta \leq \pi/3\}$. By symmetry, the total area is twice the area of D :

$$\begin{aligned} 2A(D) &= 2 \iint_D dA = 2 \int_0^{\pi/3} \int_1^{2 \cos \theta} r \, dr \, d\theta = 2 \int_0^{\pi/3} \left[\frac{1}{2} r^2 \right]_{r=1}^{r=2 \cos \theta} d\theta \\ &= \int_0^{\pi/3} (4 \cos^2 \theta - 1) \, d\theta = \int_0^{\pi/3} \left[4 \cdot \frac{1}{2} (1 + \cos 2\theta) - 1 \right] d\theta \\ &= \int_0^{\pi/3} (1 + 2 \cos 2\theta) \, d\theta = [\theta + \sin 2\theta]_0^{\pi/3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \end{aligned}$$

