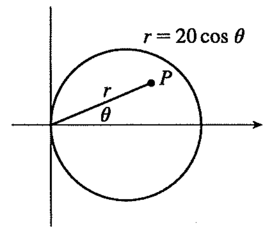
HW #15, SECTION 15.5 SOLUTIONS

35. (a) If $f(P, A)$ is the probability that an individual at A will be infected by an individual at P, and $k\,dA$ is the number of infected individuals in an element of area dA, then $f(P, A)k\,dA$ is the number of infections that should result from exposure of the individual at A to infected people in the element of area dA. Integration over D gives the number of infections of the person at A due to all the infected people in D. In rectangular coordinates (with the origin at the city's center), the exposure of a person at A is

$$E = \iint_D kf(P, A)\,dA = k\iint_D \tfrac{1}{20}\,[20 - d(P, A)]\,dA = k\iint_D \left[1 - \tfrac{1}{20}\sqrt{(x - x_0)^2 + (y - y_0)^2}\right] dA$$

(b) If $A = (0, 0)$, then



$$E = k\iint_D \left[1 - \tfrac{1}{20}\sqrt{x^2 + y^2}\right] dA$$

$$= k\int_0^{2\pi}\int_0^{10} \left(1 - \tfrac{1}{20}r\right) r\,dr\,d\theta = 2\pi k\left[\tfrac{1}{2}r^2 - \tfrac{1}{60}r^3\right]_0^{10}$$

$$= 2\pi k\left(50 - \tfrac{50}{3}\right) = \tfrac{200}{3}\pi k \approx 209k$$

For A at the edge of the city, it is convenient to use a polar coordinate system centered at A. Then the polar equation for the circular boundary of the city becomes $r = 20\cos\theta$ instead of $r = 10$, and the distance from A to a point P in the city is again r (see the figure). So

$$E = k\int_{-\pi/2}^{\pi/2}\int_0^{20\cos\theta} \left(1 - \tfrac{1}{20}r\right) r\,dr\,d\theta = k\int_{-\pi/2}^{\pi/2} \left[\tfrac{1}{2}r^2 - \tfrac{1}{60}r^3\right]_{r=0}^{r=20\cos\theta} d\theta$$

$$= k\textstyle\int_{-\pi/2}^{\pi/2} \left(200\cos^2\theta - \tfrac{400}{3}\cos^3\theta\right) d\theta = 200k\int_{-\pi/2}^{\pi/2} \left[\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta - \tfrac{2}{3}\left(1 - \sin^2\theta\right)\cos\theta\right] d\theta$$

$$= 200k\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta - \tfrac{2}{3}\sin\theta + \tfrac{2}{3}\cdot\tfrac{1}{3}\sin^3\theta\right]_{-\pi/2}^{\pi/2} = 200k\left[\tfrac{\pi}{4} + 0 - \tfrac{2}{3} + \tfrac{2}{9} + \tfrac{\pi}{4} + 0 - \tfrac{2}{3} + \tfrac{2}{9}\right]$$

$$= 200k\left(\tfrac{\pi}{2} - \tfrac{8}{9}\right) \approx 136k$$

Therefore the risk of infection is much lower at the edge of the city than in the middle, so it is better to live at the edge.

15.5 Surface Area

1. Here $z = f(x, y) = 10 + x + y^2$ and D is the triangle with vertices $(0, 0)$, $(0, -2)$, and $(2, -2)$. By Formula 2, the area of the surface is

$$A(S) = \textstyle\iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\,dA = \int_{-2}^{0}\int_0^{-y} \sqrt{1^2 + (2y)^2 + 1}\,dx\,dy$$

$$= \textstyle\int_{-2}^{0}\int_0^{-y} \sqrt{2 + 4y^2}\,dx\,dy = \int_{-2}^{0} \sqrt{2 + 4y^2}\,[x]_{x=0}^{x=-y}\,dy = -\int_{-2}^{0} y\sqrt{2 + 4y^2}\,dy$$

$$= -\frac{1}{8}\cdot\left[\frac{2}{3}(2 + 4y^2)^{3/2}\right]_{-2}^{0} = \frac{18^{3/2} - 2^{3/2}}{12} = \frac{54\sqrt{2} - 2\sqrt{2}}{12} = \frac{13\sqrt{2}}{3}$$

2. Here $z = f(x, y) = 3 + xy$ and D is the circle $x^2 + y^2 \le 1$. By Formula 2, the area of the surface is

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\,dA = \iint_{x^2+y^2\le 1} \sqrt{y^2 + x^2 + 1}\,dA \qquad \text{[Switch to polar coordinates]}$$

$$= \textstyle\int_0^{2\pi}\int_0^1 \sqrt{r^2 + 1}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 \sqrt{r^2 + 1}\,r\,dr = 2\pi\cdot\tfrac{1}{2}\left[\tfrac{2}{3}(r^2 + 1)^{3/2}\right]_{r=0}^{r=1} = \tfrac{2\pi}{3}\left(2^{3/2} - 1\right)$$

3. Here $z = f(x, y) = 5x + 3y + 6$ and D is the rectangle $[1, 4] \times [2, 6]$. By Formula 2, the area of the surface is

$$\begin{aligned} A(S) &= \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA = \iint_D \sqrt{5^2 + 3^2 + 1}\, dA = \sqrt{35} \iint_D dA \\ &= \sqrt{35}\, A(D) = \sqrt{35}\,(4-1)(6-2) = 12\sqrt{35} \end{aligned}$$

4. Here $z = f(x, y) = \frac{1}{2} - 3x - 2y$ and D is the disk $x^2 + y^2 \le 25$. By Formula 2, the area of the surface is

$$A(S) = \iint_D \sqrt{(-3)^2 + (-2)^2 + 1}\, dA = \sqrt{14} \iint_D dA = \sqrt{14}\, A(D) = \sqrt{14}\,(\pi \cdot 5^2) = 25\sqrt{14}\,\pi$$

5. The surface S is given by $z = f(x, y) = 6 - 3x - 2y$ which intersects the xy-plane in the line $3x + 2y = 6$, so D is the triangular region given by $\{(x, y) \mid 0 \le x \le 2, 0 \le y \le 3 - \frac{3}{2}x\}$. By Formula 2, the surface area of S is

$$A(S) = \iint_D \sqrt{(-3)^2 + (-2)^2 + 1}\, dA = \sqrt{14} \iint_D dA = \sqrt{14}\, A(D) = \sqrt{14}\left(\tfrac{1}{2} \cdot 2 \cdot 3\right) = 3\sqrt{14}$$

6. $z = f(x, y) = \frac{1}{4}x^2 - \frac{1}{2}y + \frac{5}{4}$, and D is the triangular region given by $\{(x, y) \mid 0 \le x \le 2,\ 0 \le y \le 2x\}$. By Formula 2,

$$\begin{aligned} A(S) &= \iint_D \sqrt{\left(\tfrac{1}{2}x\right)^2 + \left(-\tfrac{1}{2}\right)^2 + 1}\, dA = \int_0^2 \int_0^{2x} \sqrt{\tfrac{1}{4}x^2 + \tfrac{5}{4}}\, dy\, dx = \int_0^2 \tfrac{1}{2}\sqrt{x^2+5}\, \big[\, y\, \big]_{y=0}^{y=2x}\, dx \\ &= \tfrac{1}{2}\int_0^2 2x\sqrt{x^2+5}\, dx = \tfrac{1}{2} \cdot \tfrac{2}{3}(x^2+5)^{3/2}\Big]_0^2 = \tfrac{1}{3}(9^{3/2} - 5^{3/2}) = 9 - \tfrac{5}{3}\sqrt{5} \end{aligned}$$

7. The paraboloid intersects the plane $z = -2$ when $1 - x^2 - y^2 = -2 \iff x^2 + y^2 = 3$, so $D = \{(x, y) \mid x^2 + y^2 \le 3\}$. Here $z = f(x, y) = 1 - x^2 - y^2 \Rightarrow f_x = -2x$, $f_y = -2y$ and

$$\begin{aligned} A(S) &= \iint_D \sqrt{(-2x)^2 + (-2y)^2 + 1}\, dA = \iint_D \sqrt{4(x^2+y^2)+1}\, dA = \int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{4r^2+1}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} r\sqrt{4r^2+1}\, dr = \big[\,\theta\,\big]_0^{2\pi} \left[\tfrac{1}{12}(4r^2+1)^{3/2}\right]_0^{\sqrt{3}} = 2\pi \cdot \tfrac{1}{12}\left(13^{3/2} - 1\right) = \tfrac{\pi}{6}\left(13\sqrt{13} - 1\right) \end{aligned}$$

8. $x^2 + z^2 = 4 \Rightarrow z = \sqrt{4 - x^2}$ (since $z \ge 0$), so $f_x = -x(4-x^2)^{-1/2}$, $f_y = 0$ and

$$\begin{aligned} A(S) &= \int_0^1 \int_0^1 \sqrt{[-x(4-x^2)^{-1/2}]^2 + 0^2 + 1}\ dy\, dx = \int_0^1 \int_0^1 \sqrt{\frac{x^2}{4-x^2} + 1}\ dy\, dx \\ &= \int_0^1 \frac{2}{\sqrt{4-x^2}}\, dx \int_0^1 dy = \left[2\sin^{-1}\frac{x}{2}\right]_0^1 \big[\,y\,\big]_0^1 = \left(2 \cdot \tfrac{\pi}{6} - 0\right)(1) = \tfrac{\pi}{3} \end{aligned}$$

9. $z = f(x, y) = y^2 - x^2$ with $1 \le x^2 + y^2 \le 4$. Then

$$\begin{aligned} A(S) &= \iint_D \sqrt{4x^2 + 4y^2 + 1}\, dA = \int_0^{2\pi}\int_1^2 \sqrt{4r^2+1}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_1^2 r\sqrt{4r^2+1}\, dr \\ &= \big[\,\theta\,\big]_0^{2\pi} \left[\tfrac{1}{12}(4r^2+1)^{3/2}\right]_1^2 = \tfrac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \end{aligned}$$

10. $z = f(x, y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ and $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. Then $f_x = x^{1/2}$, $f_y = y^{1/2}$ and

$$\begin{aligned} A(S) &= \iint_D \sqrt{\left(\sqrt{x}\right)^2 + \left(\sqrt{y}\right)^2 + 1}\, dA = \int_0^1 \int_0^1 \sqrt{x+y+1}\, dy\, dx = \int_0^1 \left[\tfrac{2}{3}(x+y+1)^{3/2}\right]_{y=0}^{y=1} dx \\ &= \tfrac{2}{3}\int_0^1 \left[(x+2)^{3/2} - (x+1)^{3/2}\right] dx = \tfrac{2}{3}\left[\tfrac{2}{5}(x+2)^{5/2} - \tfrac{2}{5}(x+1)^{5/2}\right]_0^1 \\ &= \tfrac{4}{15}(3^{5/2} - 2^{5/2} - 2^{5/2} + 1) = \tfrac{4}{15}(3^{5/2} - 2^{7/2} + 1) \end{aligned}$$

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11. $z = f(x, y) = xy$ with $x^2 + y^2 \le 1$, so $f_x = y$, $f_y = x \Rightarrow$

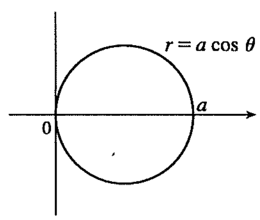
$$A(S) = \iint_D \sqrt{y^2 + x^2 + 1}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \left[\tfrac{1}{3}(r^2 + 1)^{3/2} \right]_{r=0}^{r=1} d\theta$$
$$= \int_0^{2\pi} \tfrac{1}{3}\left(2\sqrt{2} - 1\right) d\theta = \tfrac{2\pi}{3}\left(2\sqrt{2} - 1\right)$$

12. Given the sphere $x^2 + y^2 + z^2 = 4$, when $z = 1$, we get $x^2 + y^2 = 3$ so $D = \{(x, y) \mid x^2 + y^2 \le 3\}$ and $z = f(x, y) = \sqrt{4 - x^2 - y^2}$. Thus

$$A(S) = \iint_D \sqrt{[(-x)(4 - x^2 - y^2)^{-1/2}]^2 + [(-y)(4 - x^2 - y^2)^{-1/2}]^2 + 1}\, dA$$
$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{\frac{r^2}{4 - r^2} + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{\frac{r^2 + 4 - r^2}{4 - r^2}}\, r\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r}{\sqrt{4 - r^2}}\, dr\, d\theta$$
$$= \int_0^{2\pi} \left[-2(4 - r^2)^{1/2} \right]_{r=0}^{r=\sqrt{3}} d\theta = \int_0^{2\pi} (-2 + 4)\, d\theta = 2\theta \Big]_0^{2\pi} = 4\pi$$

13. $z = \sqrt{a^2 - x^2 - y^2}$, $z_x = -x(a^2 - x^2 - y^2)^{-1/2}$, $z_y = -y(a^2 - x^2 - y^2)^{-1/2}$,

$$A(S) = \iint_D \sqrt{\frac{x^2 + y^2}{a^2 - x^2 - y^2} + 1}\, dA$$
$$= \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\theta} \sqrt{\frac{r^2}{a^2 - r^2} + 1}\, r\, dr\, d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\theta} \frac{ar}{\sqrt{a^2 - r^2}}\, dr\, d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \left[-a\sqrt{a^2 - r^2} \right]_{r=0}^{r=a\cos\theta} d\theta$$
$$= \int_{-\pi/2}^{\pi/2} -a\left(\sqrt{a^2 - a^2\cos^2\theta} - a \right) d\theta = 2a^2 \int_0^{\pi/2} \left(1 - \sqrt{1 - \cos^2\theta} \right) d\theta$$
$$= 2a^2 \int_0^{\pi/2} d\theta - 2a^2 \int_0^{\pi/2} \sqrt{\sin^2\theta}\, d\theta = a^2\pi - 2a^2 \int_0^{\pi/2} \sin\theta\, d\theta = a^2(\pi - 2)$$



14. To find the region D: $z = x^2 + y^2$ implies $z + z^2 = 4z$ or $z^2 - 3z = 0$. Thus $z = 0$ or $z = 3$ are the planes where the surfaces intersect. But $x^2 + y^2 + z^2 = 4z$ implies $x^2 + y^2 + (z - 2)^2 = 4$, so $z = 3$ intersects the upper hemisphere. Thus $(z - 2)^2 = 4 - x^2 - y^2$ or $z = 2 + \sqrt{4 - x^2 - y^2}$. Therefore D is the region inside the circle $x^2 + y^2 + (3 - 2)^2 = 4$, that is, $D = \{(x, y) \mid x^2 + y^2 \le 3\}$.

$$A(S) = \iint_D \sqrt{[(-x)(4 - x^2 - y^2)^{-1/2}]^2 + [(-y)(4 - x^2 - y^2)^{-1/2}]^2 + 1}\, dA$$
$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{\frac{r^2}{4 - r^2} + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r\, dr}{\sqrt{4 - r^2}}\, d\theta = \int_0^{2\pi} \left[-2(4 - r^2)^{1/2} \right]_{r=0}^{r=\sqrt{3}} d\theta$$
$$= \int_0^{2\pi} (-2 + 4)\, d\theta = 2\theta \Big]_0^{2\pi} = 4\pi$$