HW #3, Sec 11.1 Solutions

11 ☐ SEQUENCES, SERIES, AND POWER SERIES

11.1 Sequences

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.

(b) The terms a_n approach 8 as n becomes large. In fact, we can make a_n as close to 8 as we like by taking n sufficiently large.

(c) The terms a_n become large as n becomes large. In fact, we can make a_n as large as we like by taking n sufficiently large.

2. (a) From Definition 1, a convergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ exists. Examples: $\{1/n\}$, $\{1/2^n\}$

(b) A divergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ *does not* exist. Examples: $\{n\}$, $\{\sin n\}$

3. $a_n = n^3 - 1$, so the sequence is $\{1^3-1, 2^3-1, 3^3-1, 4^3-1, 5^3-1, \ldots\} = \{0, 7, 26, 63, 124, \ldots\}$.

4. $a_n = \dfrac{1}{3^n+1}$, so the sequence is $\left\{\dfrac{1}{3^1+1}, \dfrac{1}{3^2+1}, \dfrac{1}{3^3+1}, \dfrac{1}{3^4+1}, \dfrac{1}{3^5+1}, \ldots\right\} = \left\{\dfrac{1}{4}, \dfrac{1}{10}, \dfrac{1}{28}, \dfrac{1}{82}, \dfrac{1}{244}, \ldots\right\}$.

5. $\{2^n + n\}_{n=2}^{\infty}$, so the sequence is $\{2^2+2, 2^3+3, 2^4+4, 2^5+5, 2^6+6, \ldots\} = \{6, 11, 20, 37, 70, \ldots\}$.

6. $\left\{\dfrac{n^2-1}{n^2+1}\right\}_{n=3}^{\infty}$, so the sequence is

$$\left\{\frac{3^2-1}{3^2+1}, \frac{4^2-1}{4^2+1}, \frac{5^2-1}{5^2+1}, \frac{6^2-1}{6^2+1}, \frac{7^2-1}{7^2+1}, \ldots\right\} = \left\{\frac{8}{10}, \frac{15}{17}, \frac{24}{26}, \frac{35}{37}, \frac{48}{50}, \ldots\right\}.$$

7. $a_n = \dfrac{(-1)^{n-1}}{n^2}$, so the sequence is

$$\left\{\frac{(-1)^{1-1}}{1^2}, \frac{(-1)^{2-1}}{2^2}, \frac{(-1)^{3-1}}{3^2}, \frac{(-1)^{4-1}}{4^2}, \frac{(-1)^{5-1}}{5^2}, \ldots\right\} = \left\{1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots\right\}.$$

8. $a_n = \dfrac{(-1)^n}{4^n}$, so the sequence is $\left\{\dfrac{(-1)^1}{4^1}, \dfrac{(-1)^2}{4^2}, \dfrac{(-1)^3}{4^3}, \dfrac{(-1)^4}{4^4}, \dfrac{(-1)^5}{4^5}, \ldots\right\} = \left\{-\dfrac{1}{4}, \dfrac{1}{16}, -\dfrac{1}{64}, \dfrac{1}{256}, -\dfrac{1}{1024}, \ldots\right\}$.

9. $a_n = \cos n\pi$, so the sequence is $\{\cos\pi, \cos 2\pi, \cos 3\pi, \cos 4\pi, \cos 5\pi, \ldots\} = \{-1, 1, -1, 1, -1, \ldots\}$.

10. $a_n = 1 + (-1)^n$, so the sequence is $\{1-1, 1+1, 1-1, 1+1, 1-1, \ldots\} = \{0, 2, 0, 2, 0, \ldots\}$.

11. $a_n = \dfrac{(-2)^n}{(n+1)!}$, so the sequence is

$$\left\{\frac{(-2)^1}{2!}, \frac{(-2)^2}{3!}, \frac{(-2)^3}{4!}, \frac{(-2)^4}{5!}, \frac{(-2)^5}{6!}, \ldots\right\} = \left\{-\frac{2}{2}, \frac{4}{6}, -\frac{8}{24}, \frac{16}{120}, -\frac{32}{720}, \ldots\right\} = \left\{-1, \frac{2}{3}, -\frac{1}{3}, \frac{2}{15}, -\frac{2}{45}, \ldots\right\}.$$

12. $a_n = \dfrac{2n+1}{n!+1}$, so the sequence is $\left\{\dfrac{2+1}{1+1}, \dfrac{4+1}{2+1}, \dfrac{6+1}{6+1}, \dfrac{8+1}{24+1}, \dfrac{10+1}{120+1}, \ldots\right\} = \left\{\dfrac{3}{2}, \dfrac{5}{3}, \dfrac{7}{7}, \dfrac{9}{25}, \dfrac{11}{121}, \ldots\right\}$.

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13. $a_1 = 1, a_{n+1} = 2a_n + 1$. $a_2 = 2a_1 + 1 = 2 \cdot 1 + 1 = 3$. $a_3 = 2a_2 + 1 = 2 \cdot 3 + 1 = 7$. $a_4 = 2a_3 + 1 = 2 \cdot 7 + 1 = 15$. $a_5 = 2a_4 + 1 = 2 \cdot 15 + 1 = 31$. The sequence is $\{1, 3, 7, 15, 31, \ldots\}$.

14. $a_1 = 6, a_{n+1} = \dfrac{a_n}{n}$. $a_2 = \dfrac{a_1}{1} = \dfrac{6}{1} = 6$. $a_3 = \dfrac{a_2}{2} = \dfrac{6}{2} = 3$. $a_4 = \dfrac{a_3}{3} = \dfrac{3}{3} = 1$. $a_5 = \dfrac{a_4}{4} = \dfrac{1}{4}$.

The sequence is $\{6, 6, 3, 1, \frac{1}{4}, \ldots\}$.

15. $a_1 = 2, a_{n+1} = \dfrac{a_n}{1 + a_n}$. $a_2 = \dfrac{a_1}{1 + a_1} = \dfrac{2}{1+2} = \dfrac{2}{3}$. $a_3 = \dfrac{a_2}{1 + a_2} = \dfrac{2/3}{1 + 2/3} = \dfrac{2}{5}$. $a_4 = \dfrac{a_3}{1 + a_3} = \dfrac{2/5}{1 + 2/5} = \dfrac{2}{7}$.

$a_5 = \dfrac{a_4}{1 + a_4} = \dfrac{2/7}{1 + 2/7} = \dfrac{2}{9}$. The sequence is $\{2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \ldots\}$.

16. $a_1 = 2, a_2 = 1, a_{n+1} = a_n - a_{n-1}$. Each term is defined in term of the two preceding terms.

$a_3 = a_2 - a_1 = 1 - 2 = -1$. $a_4 = a_3 - a_2 = -1 - 1 = -2$. $a_5 = a_4 - a_3 = -2 - (-1) = -1$.

The sequence is $\{2, 1, -1, -2, -1, \ldots\}$.

17. $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots\}$. The denominator is two times the number of the term, n, so $a_n = \dfrac{1}{2n}$.

18. $\{4, -1, \frac{1}{4}, -\frac{1}{16}, \frac{1}{64}, \ldots\}$. The first term is 4 and each term is $-\frac{1}{4}$ times the preceding one, so $a_n = 4\left(-\frac{1}{4}\right)^{n-1}$.

19. $\{-3, 2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27}, \ldots\}$. The first term is -3 and each term is $-\frac{2}{3}$ times the preceding one, so $a_n = -3\left(-\frac{2}{3}\right)^{n-1}$.

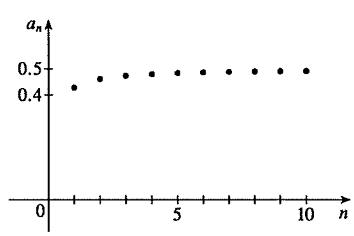
20. $\{5, 8, 11, 14, 17, \ldots\}$. Each term is larger than the preceding term by 3, so $a_n = a_1 + d(n-1) = 5 + 3(n-1) = 3n + 2$.

21. $\{\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \ldots\}$. The numerator of the nth term is n^2 and its denominator is $n + 1$. Including the alternating signs, we get $a_n = (-1)^{n+1} \dfrac{n^2}{n+1}$.

22. $\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$. Two possibilities are $a_n = \sin \dfrac{n\pi}{2}$ and $a_n = \cos \dfrac{(n-1)\pi}{2}$.

23.

n	$a_n = \dfrac{3n}{1+6n}$
1	0.4286
2	0.4615
3	0.4737
4	0.4800
5	0.4839
6	0.4865
7	0.4884
8	0.4898
9	0.4909
10	0.4918

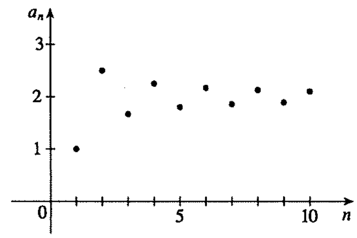


It appears that $\lim\limits_{n\to\infty} a_n = 0.5$.

$$\lim_{n\to\infty} \frac{3n}{1+6n} = \lim_{n\to\infty} \frac{(3n)/n}{(1+6n)/n} = \lim_{n\to\infty} \frac{3}{1/n + 6} = \frac{3}{6} = \frac{1}{2}$$

24.

n	$a_n = 2 + \frac{(-1)^n}{n}$
1	1.0000
2	2.5000
3	1.6667
4	2.2500
5	1.8000
6	2.1667
7	1.8571
8	2.1250
9	1.8889
10	2.1000



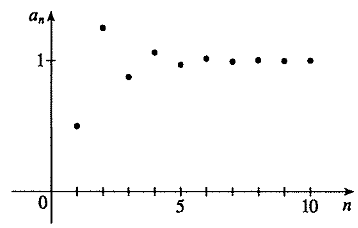
It appears that $\lim_{n\to\infty} a_n = 2$.

$\lim_{n\to\infty}\left(2 + \frac{(-1)^n}{n}\right) = \lim_{n\to\infty} 2 + \lim_{n\to\infty}\frac{(-1)^n}{n} = 2 + 0 = 2$ since $\lim_{n\to\infty}\frac{1}{n} = 0$

and by Theorem 6, $\lim_{n\to\infty}\frac{(-1)^n}{n} = 0$.

25.

n	$a_n = 1 + \left(-\frac{1}{2}\right)^n$
1	0.5000
2	1.2500
3	0.8750
4	1.0625
5	0.9688
6	1.0156
7	0.9922
8	1.0039
9	0.9980
10	1.0010



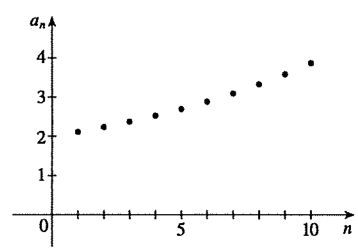
It appears that $\lim_{n\to\infty} a_n = 1$.

$\lim_{n\to\infty}\left(1 + \left(-\frac{1}{2}\right)^n\right) = \lim_{n\to\infty} 1 + \lim_{n\to\infty}\left(-\frac{1}{2}\right)^n = 1 + 0 = 1$ since

$\lim_{n\to\infty}\left(-\frac{1}{2}\right)^n = 0$ by (9).

26.

n	$a_n = 1 + \frac{10^n}{9^n}$
1	2.1111
2	2.2346
3	2.3717
4	2.5242
5	2.6935
6	2.8817
7	3.0908
8	3.3231
9	3.5812
10	3.8680



It appears that the sequence does not have a limit.

$\lim_{n\to\infty}\frac{10^n}{9^n} = \lim_{n\to\infty}\left(\frac{10}{9}\right)^n$, which diverges by (9) since $\frac{10}{9} > 1$.

27. $a_n = \frac{5}{n+2} = \frac{5/n}{(n+2)/n} = \frac{5/n}{1+2/n}$, so $a_n \to \frac{0}{1+0} = 0$ as $n \to \infty$. Converges

28. $a_n = 5\sqrt{n+2}$, so $a_n \to \infty$ as $n \to \infty$ since $\lim_{n\to\infty}\sqrt{n+2} = \infty$. Diverges

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29. $a_n = \dfrac{4n^2 - 3n}{2n^2 + 1} = \dfrac{(4n^2 - 3n)/n^2}{(2n^2 + 1)/n^2} = \dfrac{4 - 3/n}{2 + 1/n^2}$, so $a_n \to \dfrac{4 - 0}{2 + 0} = 2$ as $n \to \infty$. Converges

30. $a_n = \dfrac{4n^2 - 3n}{2n + 1} = \dfrac{(4n^2 - 3n)/n}{(2n + 1)/n} = \dfrac{4n - 3}{2 + 1/n}$, so $a_n \to \infty$ as $n \to \infty$ since $\lim\limits_{n \to \infty}(4n - 3) = \infty$ and $\lim\limits_{n \to \infty}\left(2 + \dfrac{1}{n}\right) = 2$. Diverges

31. $a_n = \dfrac{n^4}{n^3 - 2n} = \dfrac{n^4/n^3}{(n^3 - 2n)/n^3} = \dfrac{n}{1 - 2/n^2}$, so $a_n \to \infty$ as $n \to \infty$ since $\lim\limits_{n \to \infty} n = \infty$ and $\lim\limits_{n \to \infty}\left(1 - \dfrac{2}{n^2}\right) = 1 - 0 = 1$. Diverges

32. $a_n = 2 + (0.86)^n \to 2 + 0 = 2$ as $n \to \infty$ since $\lim\limits_{n \to \infty}(0.86)^n = 0$ by (9) with $r = 0.86$. Converges

33. $a_n = 3^n 7^{-n} = \dfrac{3^n}{7^n} = \left(\dfrac{3}{7}\right)^n$, so $\lim\limits_{n \to \infty} a_n = 0$ by (9) with $r = \dfrac{3}{7}$. Converges

34. $a_n = \dfrac{3\sqrt{n}}{\sqrt{n} + 2} = \dfrac{3\sqrt{n}/\sqrt{n}}{(\sqrt{n} + 2)/\sqrt{n}} = \dfrac{3}{1 + 2/\sqrt{n}} \to \dfrac{3}{1 + 0} = 3$ as $n \to \infty$. Converges

35. Because the natural exponential function is continuous at 0, Theorem 7 enables us to write $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} e^{-1/\sqrt{n}} = e^{\lim_{n \to \infty}(-1/\sqrt{n})} = e^0 = 1$. Converges

36. $a_n = \dfrac{4^n}{1 + 9^n} = \dfrac{4^n/9^n}{(1 + 9^n)/9^n} = \dfrac{(4/9)^n}{(1/9)^n + 1} \to \dfrac{0}{0 + 1} = 0$ as $n \to \infty$ since $\lim\limits_{n \to \infty}\left(\dfrac{4}{9}\right)^n = 0$ and $\lim\limits_{n \to \infty}\left(\dfrac{1}{9}\right)^n = 0$ by (9). Converges

37. $a_n = \sqrt{\dfrac{1 + 4n^2}{1 + n^2}} = \sqrt{\dfrac{(1 + 4n^2)/n^2}{(1 + n^2)/n^2}} = \sqrt{\dfrac{(1/n^2) + 4}{(1/n^2) + 1}} \to \sqrt{4} = 2$ as $n \to \infty$ since $\lim\limits_{n \to \infty}(1/n^2) = 0$. Converges

38. $a_n = \cos\left(\dfrac{n\pi}{n + 1}\right) = \cos\left(\dfrac{n\pi/n}{(n + 1)/n}\right) = \cos\left(\dfrac{\pi}{1 + 1/n}\right)$, so $a_n \to \cos\pi = -1$ as $n \to \infty$ since $\lim\limits_{n \to \infty} 1/n = 0$. Converges

39. $a_n = \dfrac{n^2}{\sqrt{n^3 + 4n}} = \dfrac{n^2/\sqrt{n^3}}{\sqrt{n^3 + 4n}/\sqrt{n^3}} = \dfrac{\sqrt{n}}{\sqrt{1 + 4/n^2}}$, so $a_n \to \infty$ as $n \to \infty$ since $\lim\limits_{n \to \infty}\sqrt{n} = \infty$ and $\lim\limits_{n \to \infty}\sqrt{1 + 4/n^2} = 1$. Diverges

40. If $b_n = \dfrac{2n}{n + 2}$, then $\lim\limits_{n \to \infty} b_n = \lim\limits_{n \to \infty}\dfrac{(2n)/n}{(n + 2)/n} = \lim\limits_{n \to \infty}\dfrac{2}{1 + 2/n} = \dfrac{2}{1} = 2$. Since the natural exponential function is continuous at 2, by Theorem 7, $\lim\limits_{n \to \infty} e^{2n/(n+2)} = e^{\lim_{n \to \infty} b_n} = e^2$. Converges

41. $\lim\limits_{n \to \infty}|a_n| = \lim\limits_{n \to \infty}\left|\dfrac{(-1)^n}{2\sqrt{n}}\right| = \dfrac{1}{2}\lim\limits_{n \to \infty}\dfrac{1}{n^{1/2}} = \dfrac{1}{2}(0) = 0$, so $\lim\limits_{n \to \infty} a_n = 0$ by (6). Converges

42. $\lim_{n\to\infty}\frac{n}{n+\sqrt{n}}=\lim_{n\to\infty}\frac{n/n}{(n+\sqrt{n})/n}=\lim_{n\to\infty}\frac{1}{1+1/\sqrt{n}}=\frac{1}{1+0}=1$. Thus, $a_n=\frac{(-1)^{n+1}n}{n+\sqrt{n}}$ has odd-numbered terms that approach 1 and even-numbered terms that approach -1 as $n\to\infty$, and hence, the sequence $\{a_n\}$ is divergent.

43. $a_n=\frac{(2n-1)!}{(2n+1)!}=\frac{(2n-1)!}{(2n+1)(2n)(2n-1)!}=\frac{1}{(2n+1)(2n)}\to 0$ as $n\to\infty$. Converges

44. $a_n=\frac{\ln n}{\ln(2n)}=\frac{\ln n}{\ln 2+\ln n}=\frac{(\ln n)/\ln n}{(\ln 2+\ln n)/\ln n}=\frac{1}{\frac{\ln 2}{\ln n}+1}\to\frac{1}{0+1}=1$ as $n\to\infty$. Converges

45. $a_n=\sin n$. This sequence diverges since the terms don't approach any particular real number as $n\to\infty$. The terms take on values between -1 and 1. Diverges

46. $a_n=\frac{\tan^{-1}n}{n}$. $\lim_{n\to\infty}\tan^{-1}n=\lim_{x\to\infty}\tan^{-1}x=\frac{\pi}{2}$ by (4), so $\lim_{n\to\infty}a_n=0$. Converges

47. $a_n=n^2e^{-n}=\frac{n^2}{e^n}$. Since $\lim_{x\to\infty}\frac{x^2}{e^x}\overset{H}{=}\lim_{x\to\infty}\frac{2x}{e^x}\overset{H}{=}\lim_{x\to\infty}\frac{2}{e^x}=0$, it follows from Theorem 4 that $\lim_{n\to\infty}a_n=0$. Converges

48. $a_n=\ln(n+1)-\ln n=\ln\left(\frac{n+1}{n}\right)=\ln\left(1+\frac{1}{n}\right)\to\ln(1)=0$ as $n\to\infty$ because ln is continuous. Converges

49. $0\le\frac{\cos^2 n}{2^n}\le\frac{1}{2^n}$ [since $0\le\cos^2 n\le 1$], so since $\lim_{n\to\infty}\frac{1}{2^n}=0$, $\left\{\frac{\cos^2 n}{2^n}\right\}$ converges to 0 by the Squeeze Theorem.

50. $a_n=\sqrt[n]{2^{1+3n}}=(2^{1+3n})^{1/n}=(2^1 2^{3n})^{1/n}=2^{1/n}2^3=8\cdot 2^{1/n}$, so

$\lim_{n\to\infty}a_n=8\lim_{n\to\infty}2^{1/n}=8\cdot 2^{\lim_{n\to\infty}(1/n)}=8\cdot 2^0=8$ by Theorem 7, since the function $f(x)=2^x$ is continuous at 0.

Converges

51. $a_n=n\sin(1/n)=\frac{\sin(1/n)}{1/n}$. Since $\lim_{x\to\infty}\frac{\sin(1/x)}{1/x}=\lim_{t\to 0^+}\frac{\sin t}{t}$ [where $t=1/x$] $=1$, it follows from Theorem 4 that $\{a_n\}$ converges to 1.

52. $a_n=2^{-n}\cos n\pi$. $0\le\left|\frac{\cos n\pi}{2^n}\right|\le\frac{1}{2^n}=\left(\frac{1}{2}\right)^n$, so $\lim_{n\to\infty}|a_n|=0$ by (9), and $\lim_{n\to\infty}a_n=0$ by (6). Converges

53. $y=\left(1+\frac{2}{x}\right)^x \Rightarrow \ln y=x\ln\left(1+\frac{2}{x}\right)$, so

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln(1+2/x)}{1/x}\overset{H}{=}\lim_{x\to\infty}\frac{\left(\frac{1}{1+2/x}\right)\left(-\frac{2}{x^2}\right)}{-1/x^2}=\lim_{x\to\infty}\frac{2}{1+2/x}=2 \Rightarrow$$

$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x=\lim_{x\to\infty}e^{\ln y}=e^2$, so by Theorem 4, $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n=e^2$. Converges

54. $y=x^{1/x} \Rightarrow \ln y=\frac{1}{x}\ln x$, so $\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln x}{x}\overset{H}{=}\lim_{x\to\infty}\frac{1/x}{1}=\lim_{x\to\infty}\frac{1}{x}=0 \Rightarrow$

$\lim_{x\to\infty}x^{1/x}=\lim_{x\to\infty}e^{\ln y}=e^0=1$, so by Theorem 4, $\lim_{n\to\infty}n^{1/n}=1$. Converges