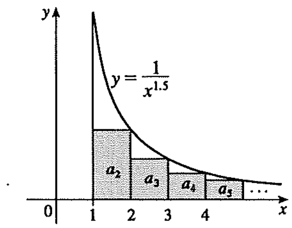
HW #3, Sec 11.3 Solutions

11.3 The Integral Test and Estimates of Sums

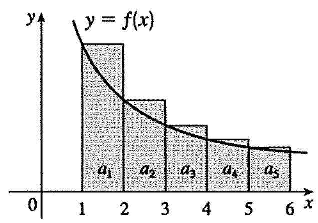
1. The picture shows that $a_2 = \frac{1}{2^{1.5}} < \int_1^2 \frac{1}{x^{1.5}}\,dx$,

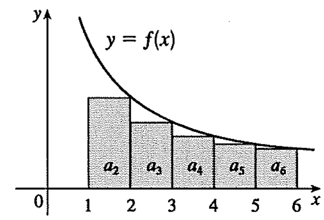
$a_3 = \frac{1}{3^{1.5}} < \int_2^3 \frac{1}{x^{1.5}}\,dx$, and so on, so $\sum_{n=2}^{\infty} \frac{1}{n^{1.5}} < \int_1^{\infty} \frac{1}{x^{1.5}}\,dx$.

The integral converges by (7.8.2) with $p = 1.5 > 1$, so the series converges.



2. From the first figure, we see that $\int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$. From the second figure, we see that $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx$. Thus, we have $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$.





3. The function $f(x) = x^{-3}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} x^{-3}\,dx = \lim_{t\to\infty} \int_1^t x^{-3}\,dx = \lim_{t\to\infty} \left[\frac{x^{-2}}{-2}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2} + \frac{1}{2}\right) = \frac{1}{2}.$$

Since this improper integral is convergent, the series $\sum_{n=1}^{\infty} n^{-3}$ is also convergent by the Integral Test.

4. The function $f(x) = x^{-0.3}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} x^{-0.3}\,dx = \lim_{t\to\infty} \int_1^t x^{-0.3}\,dx = \lim_{t\to\infty} \left[\frac{x^{0.7}}{0.7}\right]_1^t = \lim_{t\to\infty}\left(\frac{t^{0.7}}{0.7} - \frac{1}{0.7}\right) = \infty.$$

Since this improper integral is divergent, the series $\sum_{n=1}^{\infty} n^{-0.3}$ is also divergent by the Integral Test.

5. The function $f(x) = \frac{2}{5x-1}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{2}{5x-1}\,dx = \lim_{t\to\infty} \int_1^t \frac{2}{5x-1}\,dx = \lim_{t\to\infty} \left[\frac{2}{5}\ln(5x-1)\right]_1^t = \lim_{t\to\infty}\left[\frac{2}{5}\ln(5t-1) - \frac{2}{5}\ln 4\right] = \infty.$$

Since this improper integral is divergent, the series $\sum_{n=1}^{\infty} \frac{2}{5n-1}$ is also divergent by the Integral Test.

6. The function $f(x) = \frac{1}{(3x-1)^4}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{1}{(3x-1)^4}\,dx = \lim_{t\to\infty} \int_1^t (3x-1)^{-4}\,dx = \lim_{t\to\infty} \left[\frac{1}{(-3)3}(3x-1)^{-3}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{9(3t-1)^3} + \frac{1}{9\cdot 2^3}\right] = \frac{1}{72}.$$

Since this improper integral is convergent, the series $\sum_{n=1}^{\infty} \frac{1}{(3n-1)^4}$ is also convergent by the Integral Test.

Sec 11.3

7. The function $f(x) = \dfrac{x^2}{x^3+1}$ is continuous, positive, and decreasing ($\star$) on $[2,\infty)$, so the Integral Test applies.

$$\int_2^\infty \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty}\int_2^t \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty}\left[\frac{1}{3}\ln\left|x^3+1\right|\right]_2^t = \lim_{t\to\infty}\left[\frac{1}{3}\ln\left(t^3+1\right) - \frac{1}{3}\ln 9\right] = \infty.$$

Since the improper integral is divergent, the series $\sum\limits_{n=2}^{\infty}\dfrac{n^2}{n^3+1}$ is also divergent by the Integral Test.

$$(\star)\text{: } f'(x) = \frac{(x^3+1)(2x) - x^2(3x^2)}{(x^3+1)^2} = \frac{2x - x^4}{(x^3+1)^2} = -\frac{x(x^3-2)}{(x^3+1)^2} < 0 \text{ for } x \geq 2.$$

8. The function $f(x) = x^2e^{-x^3}$ is continuous, positive, and decreasing ($\star$) on $[1,\infty)$, so the Integral Test applies.

$$\int_1^\infty x^2e^{-x^3}\,dx = \lim_{t\to\infty}\int_1^t x^2e^{-x^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{3}e^{-x^3}\right]_1^t = -\frac{1}{3}\lim_{t\to\infty}\left(e^{-t^3} - e^{-1}\right) = -\frac{1}{3}\left(0 - \frac{1}{e}\right) = \frac{1}{3e}.$$

Since this improper integral is convergent, the series $\sum\limits_{n=1}^{\infty} n^2e^{-n^3}$ is also convergent by the Integral Test.

$$(\star)\text{: } f'(x) = x^2e^{-x^3}(-3x^2) + e^{-x^3}(2x) = xe^{-x^3}(-3x^3+2) = \frac{x(2-3x^3)}{e^{x^3}} < 0 \text{ for } x > 1$$

9. The function $f(x) = \dfrac{1}{x(\ln x)^3}$ is continuous, positive, and decreasing on $[2,\infty)$, so the Integral Test applies.

$$\int_2^\infty \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{du}{u^3}\quad\left[u = \ln x,\ du = \frac{dx}{x}\right]\quad = \lim_{t\to\infty}\left[-\frac{1}{2}u^{-2}\right]_{\ln 2}^{\ln t}$$
$$= \lim_{t\to\infty}\left[-\frac{1}{2(\ln t)^2} + \frac{1}{2(\ln 2)^2}\right] = 0 + \frac{1}{2(\ln 2)^2} = \frac{1}{2(\ln 2)^2}$$

Since the improper integral is convergent, the series $\sum\limits_{n=2}^{\infty}\dfrac{1}{n(\ln n)^3}$ is also convergent by the Integral Test.

10. The function $f(x) = \dfrac{\tan^{-1}x}{1+x^2}$ is continuous, positive, and decreasing ($\star$) on $[1,\infty)$, so the Integral Test applies.

$$\int_1^\infty \frac{\tan^{-1}x}{1+x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\tan^{-1}x}{1+x^2}\,dx = \lim_{t\to\infty}\int_{\tan^{-1}1}^{\tan^{-1}t} u\,du\quad\left[u = \tan^{-1}x,\ du = \frac{dx}{1+x^2}\right]$$
$$= \lim_{t\to\infty}\left[\frac{1}{2}u^2\right]_{\tan^{-1}1}^{\tan^{-1}t} = \lim_{t\to\infty}\left[\frac{1}{2}(\tan^{-1}t)^2 - \frac{1}{2}(\tan^{-1}1)^2\right] = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\left(\frac{\pi}{4}\right)^2 = \frac{3\pi^2}{32}$$

Since the improper integral is convergent, the series $\sum\limits_{n=1}^{\infty}\dfrac{\tan^{-1}n}{1+n^2}$ is also convergent by the Integral Test.

$$(\star)\text{: } f'(x) = \frac{(1+x^2)\left(\dfrac{1}{1+x^2}\right) - (\tan^{-1}x)(2x)}{(1+x^2)^2} = \frac{1 - 2x\tan^{-1}x}{(1+x^2)^2} < 0 \text{ for } 2x\tan^{-1}x > 1.$$

Both $2x$ and $\tan^{-1}x$ are increasing functions, and when $x = 1$, $2x\tan^{-1}x = 2(1)\tan^{-1}1 = 2\pi/4 > 1$, so it follows that $f'(x) < 0$ when $x \geq 1$.

11. $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^{\sqrt{2}}}$ is a p-series with $p = \sqrt{2} > 1$, so it converges by (1).

12. $\sum_{n=3}^{\infty} n^{-0.9999} = \sum_{n=3}^{\infty} \frac{1}{n^{0.9999}}$ is a p-series with $p = 0.9999 \le 1$, so it diverges by (1). The fact that the series begins with $n = 3$ is irrelevant when determining convergence.

13. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^3}$. This is a p-series with $p = 3 > 1$, so it converges by (1).

14. $\frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2n+3}$. The function $f(x) = \frac{1}{2x+3}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$\int_1^{\infty} \frac{1}{2x+3}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{2x+3}\,dx = \lim_{t\to\infty} \left[\frac{1}{2}\ln(2x+3)\right]_1^t = \lim_{t\to\infty} \left[\frac{1}{2}\ln(2t+3) - \frac{1}{2}\ln 5\right] = \infty$, so the series $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ diverges.

15. $\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \frac{1}{19} + \cdots = \sum_{n=1}^{\infty} \frac{1}{4n-1}$. The function $f(x) = \frac{1}{4x-1}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$\int_1^{\infty} \frac{1}{4x-1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{4x-1}\,dx = \lim_{t\to\infty} \left[\frac{1}{4}\ln(4x-1)\right]_1^t = \lim_{t\to\infty} \left[\frac{1}{4}\ln(4t-1) - \frac{1}{4}\ln 3\right] = \infty$, so the series $\sum_{n=1}^{\infty} \frac{1}{4n-1}$ diverges.

16. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. This is a p-series with $p = \frac{3}{2} > 1$, so it converges by (1).

17. $\sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2} = \sum_{n=1}^{\infty} \left(\frac{\sqrt{n}}{n^2} + \frac{4}{n^2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} + \sum_{n=1}^{\infty} \frac{4}{n^2}$. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series with $p = \frac{3}{2} > 1$. $\sum_{n=1}^{\infty} \frac{4}{n^2} = 4\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a constant multiple of a convergent p-series with $p = 2 > 1$, so it converges. The sum of two convergent series is convergent, so the original series is convergent.

18. The function $f(x) = \frac{\sqrt{x}}{1+x^{3/2}}$ is continuous and positive on $[1, \infty)$.

$$f'(x) = \frac{(1+x^{3/2})\left(\frac{1}{2}x^{-1/2}\right) - x^{1/2}\left(\frac{3}{2}x^{1/2}\right)}{(1+x^{3/2})^2} = \frac{\frac{1}{2}x^{-1/2} + \frac{1}{2}x - \frac{3}{2}x}{(1+x^{3/2})^2} = \frac{1-2x^{3/2}}{2\sqrt{x}(1+x^{3/2})^2} < 0 \text{ for } x \ge 1, \text{ so } f \text{ is}$$

decreasing on $[1, \infty)$, and the Integral Test applies.

$$\int_1^{\infty} \frac{\sqrt{x}}{1+x^{3/2}}\,dx = \lim_{t\to\infty} \int_1^t \frac{\sqrt{x}}{1+x^{3/2}}\,dx = \lim_{t\to\infty} \left[\frac{2}{3}\ln(1+x^{3/2})\right]_1^t \quad \left[\begin{matrix}\text{substitution} \\ \text{with } u = 1 + x^{3/2}\end{matrix}\right]$$
$$= \lim_{t\to\infty} \left[\frac{2}{3}\ln(1+t^{3/2}) - \frac{2}{3}\ln 2\right] = \infty,$$

so the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^{3/2}}$ diverges.