HW #4, Sec 11.3 Solutions (HW #4, PART I)

34. If $p \le 0$, $\lim\limits_{n\to\infty} \dfrac{\ln n}{n^p} = \infty$ and the series diverges, so assume $p > 0$. $f(x) = \dfrac{\ln x}{x^p}$ is positive and continuous and $f'(x) < 0$ for $x > e^{1/p}$, so f is eventually decreasing and we can use the Integral Test. Integration by parts gives

$$\int_1^\infty \frac{\ln x}{x^p}\,dx = \lim_{t\to\infty}\left[\frac{x^{1-p}\left[(1-p)\ln x - 1\right]}{(1-p)^2}\right]_1^t \quad (\text{for } p \ne 1) = \frac{1}{(1-p)^2}\left[\lim_{t\to\infty} t^{1-p}\left[(1-p)\ln t - 1\right] + 1\right],$$ which exists whenever $1 - p < 0 \Leftrightarrow p > 1$. Thus, $\sum\limits_{n=1}^\infty \dfrac{\ln n}{n^p}$ converges $\Leftrightarrow p > 1$.

35. Since this is a p-series with $p = x$, $\zeta(x)$ is defined when $x > 1$. Unless specified otherwise, the domain of a function f is the set of real numbers x such that the expression for $f(x)$ makes sense and defines a real number. So, in the case of a series, it's the set of real numbers x such that the series is convergent.

36. (a) $\sum\limits_{n=2}^\infty \dfrac{1}{n^2} = \sum\limits_{n=1}^\infty \dfrac{1}{n^2} - \dfrac{1}{1^2}$ [subtract a_1] $= \dfrac{\pi^2}{6} - 1$

(b) $\sum\limits_{n=3}^\infty \dfrac{1}{(n+1)^2} = \sum\limits_{n=4}^\infty \dfrac{1}{n^2} = \sum\limits_{n=1}^\infty \dfrac{1}{n^2} - \left(\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2}\right) = \dfrac{\pi^2}{6} - \dfrac{49}{36}$

(c) $\sum\limits_{n=1}^\infty \dfrac{1}{(2n)^2} = \sum\limits_{n=1}^\infty \dfrac{1}{4n^2} = \dfrac{1}{4}\sum\limits_{n=1}^\infty \dfrac{1}{n^2} = \dfrac{1}{4}\left(\dfrac{\pi^2}{6}\right) = \dfrac{\pi^2}{24}$

37. (a) $\sum\limits_{n=1}^\infty \left(\dfrac{3}{n}\right)^4 = \sum\limits_{n=1}^\infty \dfrac{81}{n^4} = 81\sum\limits_{n=1}^\infty \dfrac{1}{n^4} = 81\left(\dfrac{\pi^4}{90}\right) = \dfrac{9\pi^4}{10}$

(b) $\sum\limits_{k=5}^\infty \dfrac{1}{(k-2)^4} = \dfrac{1}{3^4} + \dfrac{1}{4^4} + \dfrac{1}{5^4} + \cdots = \sum\limits_{k=3}^\infty \dfrac{1}{k^4} = \dfrac{\pi^4}{90} - \left(\dfrac{1}{1^4} + \dfrac{1}{2^4}\right)$ [subtract a_1 and a_2] $= \dfrac{\pi^4}{90} - \dfrac{17}{16}$

38. (a) $f(x) = 1/x^4$ is positive and continuous and $f'(x) = -4/x^5$ is negative for $x > 0$, and so the Integral Test applies.

$\sum\limits_{n=1}^\infty \dfrac{1}{n^4} \approx s_{10} = \dfrac{1}{1^4} + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \cdots + \dfrac{1}{10^4} \approx 1.082037.$

$R_{10} \le \displaystyle\int_{10}^\infty \frac{1}{x^4}\,dx = \lim_{t\to\infty}\left[\frac{1}{-3x^3}\right]_{10}^t = \lim_{t\to\infty}\left(-\frac{1}{3t^3} + \frac{1}{3(10)^3}\right) = \frac{1}{3000}$, so the error is at most $0.000\overline{3}$.

(b) $s_{10} + \displaystyle\int_{11}^\infty \frac{1}{x^4}\,dx \le s \le s_{10} + \int_{10}^\infty \frac{1}{x^4}\,dx \Rightarrow s_{10} + \frac{1}{3(11)^3} \le s \le s_{10} + \frac{1}{3(10)^3} \Rightarrow$

$1.082037 + 0.000250 = 1.082287 \le s \le 1.082037 + 0.000333 = 1.082370$, so we get $s \approx 1.08233$ with error ≤ 0.00005.

(c) The estimate in part (b) is $s \approx 1.08233$ with error ≤ 0.00005. The exact value given in Exercise 37 is $\pi^4/90 \approx 1.082323$. The difference is less than 0.00001.

(d) $R_n \le \displaystyle\int_n^\infty \frac{1}{x^4}\,dx = \frac{1}{3n^3}$. So $R_n < 0.00001 \Rightarrow \dfrac{1}{3n^3} < \dfrac{1}{10^5} \Rightarrow 3n^3 > 10^5 \Rightarrow n > \sqrt[3]{(10)^5/3} \approx 32.2$, that is, for $n > 32$.

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39. (a) $f(x) = \dfrac{1}{x^2}$ is positive and continuous and $f'(x) = -\dfrac{2}{x^3}$ is negative for $x > 0$, and so the Integral Test applies.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx s_{10} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{10^2} \approx 1.549768.$$

$$R_{10} \le \int_{10}^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[\frac{-1}{x}\right]_{10}^{t} = \lim_{t\to\infty} \left(-\frac{1}{t} + \frac{1}{10}\right) = \frac{1}{10},$$ so the error is at most 0.1.

(b) $s_{10} + \displaystyle\int_{11}^{\infty} \frac{1}{x^2}\,dx \le s \le s_{10} + \int_{10}^{\infty} \frac{1}{x^2}\,dx \;\Rightarrow\; s_{10} + \frac{1}{11} \le s \le s_{10} + \frac{1}{10} \;\Rightarrow$

$1.549768 + 0.090909 = 1.640677 \le s \le 1.549768 + 0.1 = 1.649768$, so we get $s \approx 1.64522$ (the average of 1.640677 and 1.649768) with error ≤ 0.005 (the maximum of $1.649768 - 1.64522$ and $1.64522 - 1.640677$, rounded up).

(c) The estimate in part (b) is $s \approx 1.64522$ with error ≤ 0.005. The exact value given in Exercise 36 is $\pi^2/6 \approx 1.644934$. The difference is less than 0.0003.

(d) $R_n \le \displaystyle\int_{n}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{n}$. So $R_n < 0.001$ if $\dfrac{1}{n} < \dfrac{1}{1000} \;\Leftrightarrow\; n > 1000$.

40. $\displaystyle\sum_{n=1}^{\infty} ne^{-2n}$. $f(x) = xe^{-2x}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies. Using (2),

$$R_n \le \int_n^{\infty} xe^{-2x}\,dx = \lim_{t\to\infty} \left(\left[-\tfrac{1}{2}xe^{-2x}\right]_n^t + \int_n^t \tfrac{1}{2}e^{-2x}\,dx \right) \quad \left[\begin{matrix}\text{using parts with} \\ u = x,\, dv = e^{-2x}\,dx\end{matrix}\right]$$

$$= \lim_{t\to\infty} \left(\frac{-t}{2e^{2t}} + \frac{n}{2e^{2n}} - \frac{1}{4e^{2t}} + \frac{1}{4e^{2n}} \right) \overset{\text{H}}{=} 0 + \frac{n}{2e^{2n}} - 0 + \frac{1}{4e^{2n}} = \frac{2n+1}{4e^{2n}}$$

To be correct to four decimal places, we want $\dfrac{2n+1}{4e^{2n}} \le \dfrac{5}{10^5}$. This inequality is true for $n = 6$.

$$s_6 = \sum_{n=1}^{6} \frac{n}{e^{2n}} = \frac{1}{e^2} + \frac{2}{e^4} + \frac{3}{e^6} + \frac{4}{e^8} + \frac{5}{e^{10}} + \frac{6}{e^{12}} \approx 0.1810.$$

41. $\displaystyle\sum_{n=1}^{\infty} (2n+1)^{-6}$. $f(x) = 1/(2x+1)^6$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

Using (2), $R_n \le \displaystyle\int_n^{\infty} (2x+1)^{-6}\,dx = \lim_{t\to\infty} \left[\frac{-1}{10(2x+1)^5}\right]_n^t = \frac{1}{10(2n+1)^5}$. To be correct to five decimal places,

we want $\dfrac{1}{10(2n+1)^5} \le \dfrac{5}{10^6} \;\Leftrightarrow\; (2n+1)^5 \ge 20{,}000 \;\Leftrightarrow\; n \ge \tfrac{1}{2}\left(\sqrt[5]{20{,}000} - 1\right) \approx 3.12$, so use $n = 4$.

$$s_4 = \sum_{n=1}^{4} \frac{1}{(2n+1)^6} = \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} \approx 0.001\,446 \approx 0.00145.$$

42. $f(x) = \dfrac{1}{x(\ln x)^2}$ is positive and continuous and $f'(x) = -\dfrac{\ln x + 2}{x^2(\ln x)^3}$ is negative for $x > 1$, so the Integral Test applies.

Using (2), we need $0.01 > \displaystyle\int_n^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty} \left[\frac{-1}{\ln x}\right]_n^t = \frac{1}{\ln n}$. This is true for $n > e^{100}$, so we would have to add this many terms to find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ to within 0.01, which would be problematic because $e^{100} \approx 2.7 \times 10^{43}$.