HW #4, Sec 11.4 SOLUTIONS (HW #4, Parts II and IV)

(c) We have shown that $\{t_n\}$ is decreasing and that $t_n > 0$ for all n. Thus, $0 < t_n \leq t_1 = 1$, so $\{t_n\}$ is a bounded monotonic sequence, and hence converges by the Monotonic Sequence Theorem.

47. $b^{\ln n} = \left(e^{\ln b}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln b} = n^{\ln b} = \dfrac{1}{n^{-\ln b}}$. $\sum\limits_{n=1}^{\infty} b^{\ln n}$ is a p-series, which converges for all b such that $-\ln b > 1$ $\Leftrightarrow$ $\ln b < -1$ $\Leftrightarrow$ $b < e^{-1}$ $\Leftrightarrow$ $b < 1/e$ [with $b > 0$].

48. For the series $\sum\limits_{n=1}^{\infty} \left(\dfrac{c}{n} - \dfrac{1}{n+1}\right)$,

$$s_n = \sum_{i=1}^{n}\left(\frac{c}{i} - \frac{1}{i+1}\right) = \left(\frac{c}{1} - \frac{1}{2}\right) + \left(\frac{c}{2} - \frac{1}{3}\right) + \left(\frac{c}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{c}{n} - \frac{1}{n+1}\right)$$

$$= \frac{c}{1} + \frac{c-1}{2} + \frac{c-1}{3} + \frac{c-1}{4} + \cdots + \frac{c-1}{n} - \frac{1}{n+1} = c + (c-1)\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right) - \frac{1}{n+1}$$

Thus, $\sum\limits_{n=1}^{\infty}\left(\dfrac{c}{n} - \dfrac{1}{n+1}\right) = \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty}\left[c + (c-1)\sum\limits_{i=2}^{n}\dfrac{1}{i} - \dfrac{1}{n+1}\right]$. Since a constant multiple of a divergent series is divergent, the last limit exists only if $c - 1 = 0$, so the original series converges only if $c = 1$.

11.4 The Comparison Tests

1. (a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent. (See the discusssion preceding the box titled "The Limit Comparison Test.")

(b) If $a_n < b_n$ for all n, then $\sum a_n$ is convergent. [This is part (i) of the Direct Comparison Test.]

2. (a) If $a_n > b_n$ for all n, then $\sum a_n$ is divergent. [This is part (ii) of the Direct Comparison Test.]

(b) We cannot say anything about $\sum a_n$. If $a_n < b_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ could be convergent or divergent.

3. (a) $\dfrac{n}{n^3+5} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$ for all $n \geq 2$. $\sum\limits_{n=2}^{\infty}\dfrac{1}{n^2}$ converges because it is a p-series with $p = 2 > 1$, so $\sum\limits_{n=2}^{\infty}\dfrac{n}{n^3+5}$ converges by part (i) of the Direct Comparison Test.

(b) Use the Limit Comparison Test with $a_n = \dfrac{n}{n^3-5}$ and $b_n = \dfrac{1}{n^2}$:

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n}{n^3-5}\cdot\frac{n^2}{1} = \lim_{n\to\infty}\frac{n^3}{n^3\left(1-5/n^3\right)} = \lim_{n\to\infty}\frac{1}{1-5/n^3} = \frac{1}{1-0} = 1 > 0$$

Since $\sum\limits_{n=2}^{\infty}\dfrac{1}{n^2}$ is a convergent (partial) p-series [$p = 2 > 1$], the series $\sum\limits_{n=2}^{\infty}\dfrac{n}{n^3-5}$ also converges.

4. (a) $\dfrac{n^2+n}{n^3-2} > \dfrac{n^2}{n^3-2} > \dfrac{n^2}{n^3} = \dfrac{1}{n}$ for all $n \geq 2$. $\sum\limits_{n=2}^{\infty}\dfrac{1}{n}$ diverges because it is a (partial) p-series with $p = 1 \leq 1$, so $\sum\limits_{n=2}^{\infty}\dfrac{n^2+n}{n^3-2}$ diverges by part (ii) of the Direct Comparison Test.

(b) Use the Limit Comparison Test with $a_n = \dfrac{n^2 - n}{n^3 + 2}$ and $b_n = \dfrac{1}{n}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2 - n}{n^3 + 2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^3 - n^2}{n^3 + 2} = \lim_{n\to\infty} \frac{1 - 1/n}{1 + 2/n^3} = \frac{1 - 0}{1 + 0} = 1 > 0$$

Since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n}$ is a divergent (partial) p-series [$p = 1 \leq 1$], the series $\sum\limits_{n=2}^{\infty} \dfrac{n^2 - n}{n^3 + 2}$ also diverges.

5. An inequality can be used to show that a series converges if its general term can be shown to be less than or equal to the general term of a known convergent series. The only inequality that satisfies this condition is given in part (c) since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ is a convergent p-series [$p = 2 > 1$].

6. An inequality can be used to show that a series diverges if its general term can be shown to be greater than or equal to the general term of a known divergent series. The only inequality that satisfies this condition is given in part (c) since $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n} = \dfrac{1}{2} \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ is half of the harmonic series, which is divergent.

7. $\dfrac{1}{n^3 + 8} < \dfrac{1}{n^3}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3 + 8}$ converges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$, which converges because it is a p-series with $p = 3 > 1$.

8. $\dfrac{1}{\sqrt{n} - 1} > \dfrac{1}{\sqrt{n}}$ for all $n \geq 2$, so $\sum\limits_{n=2}^{\infty} \dfrac{1}{\sqrt{n} - 1}$ diverges by direct comparison with $\sum\limits_{n=2}^{\infty} \dfrac{1}{\sqrt{n}}$, which diverges because it is a p-series with $p = \frac{1}{2} \leq 1$.

9. $\dfrac{n + 1}{n\sqrt{n}} > \dfrac{n}{n\sqrt{n}} = \dfrac{1}{\sqrt{n}}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n + 1}{n\sqrt{n}}$ diverges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$, which diverges because it is a p-series with $p = \frac{1}{2} \leq 1$.

10. $\dfrac{n - 1}{n^3 + 1} < \dfrac{n}{n^3 + 1} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n - 1}{n^3 + 1}$ converges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

11. $\dfrac{9^n}{3 + 10^n} < \dfrac{9^n}{10^n} = \left(\dfrac{9}{10}\right)^n$ for all $n \geq 1$. $\sum\limits_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$ is a convergent geometric series $\left(|r| = \frac{9}{10} < 1\right)$, so $\sum\limits_{n=1}^{\infty} \dfrac{9^n}{3 + 10^n}$ converges by the Direct Comparison Test.

12. $\dfrac{6^n}{5^n - 1} > \dfrac{6^n}{5^n} = \left(\dfrac{6}{5}\right)^n$ for all $n \geq 1$. $\sum\limits_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$ is a divergent geometric series $\left(|r| = \frac{6}{5} > 1\right)$, so $\sum\limits_{n=1}^{\infty} \dfrac{6^n}{5^n - 1}$ diverges by the Direct Comparison Test.

13. For $n \geq 2$, $\ln n < n$, so $\dfrac{1}{\ln n} > \dfrac{1}{n}$. Thus, $\sum\limits_{n=2}^{\infty} \dfrac{1}{\ln n}$ diverges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$, which diverges because it is a p-series with $p = 1 \leq 1$ (the harmonic series).

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14. $\dfrac{k\sin^2 k}{1+k^3} \le \dfrac{k}{1+k^3} < \dfrac{k}{k^3} = \dfrac{1}{k^2}$ for all $k \ge 1$, so $\sum\limits_{k=1}^{\infty} \dfrac{k\sin^2 k}{1+k^3}$ converges by direct comparison with $\sum\limits_{k=1}^{\infty} \dfrac{1}{k^2}$, which converges because it is a p-series with $p = 2 > 1$.

15. $\dfrac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}} < \dfrac{\sqrt[3]{k}}{\sqrt{k^3}} = \dfrac{k^{1/3}}{k^{3/2}} = \dfrac{1}{k^{7/6}}$ for all $k \ge 1$, so $\sum\limits_{k=1}^{\infty} \dfrac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}}$ converges by direct comparison with $\sum\limits_{k=1}^{\infty} \dfrac{1}{k^{7/6}}$, which converges because it is a p-series with $p = \frac{7}{6} > 1$.

16. $\dfrac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2} < \dfrac{2k(k^2)}{k(k^2)^2} = \dfrac{2k^3}{k^5} = \dfrac{2}{k^2}$ for all $k \ge 1$, so $\sum\limits_{k=1}^{\infty} \dfrac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$ converges by direct comparison with $2\sum\limits_{k=1}^{\infty} \dfrac{1}{k^2}$, which converges because it is a constant multiple of a p-series with $p = 2 > 1$.

17. $\dfrac{1+\cos n}{e^n} < \dfrac{2}{e^n}$ for all $n \ge 1$. $\sum\limits_{n=1}^{\infty} \dfrac{2}{e^n}$ is a convergent geometric series ($|r| = \frac{1}{e} < 1$), so $\sum\limits_{n=1}^{\infty} \dfrac{1+\cos n}{e^n}$ converges by the Direct Comparison Test.

18. $\dfrac{1}{\sqrt[3]{3n^4+1}} < \dfrac{1}{\sqrt[3]{3n^4}} < \dfrac{1}{\sqrt[3]{n^4}} = \dfrac{1}{n^{4/3}}$ for all $n \ge 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt[3]{3n^4+1}}$ converges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{4/3}}$, which converges because it is a p-series with $p = \frac{4}{3} > 1$.

19. $\dfrac{4^{n+1}}{3^n-2} > \dfrac{4\cdot 4^n}{3^n} = 4\left(\dfrac{4}{3}\right)^n$ for all $n \ge 1$. $\sum\limits_{n=1}^{\infty} 4\left(\dfrac{4}{3}\right)^n = 4\sum\limits_{n=1}^{\infty}\left(\dfrac{4}{3}\right)^n$ is a divergent geometric series $\left(|r| = \frac{4}{3} > 1\right)$, so $\sum\limits_{n=1}^{\infty} \dfrac{4^{n+1}}{3^n-2}$ diverges by the Direct Comparison Test.

20. $\dfrac{1}{n^n} \le \dfrac{1}{n^2}$ for all $n \ge 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^n}$ converges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

21. Use the Limit Comparison Test with $a_n = \dfrac{1}{\sqrt{n^2+1}}$ and $b_n = \dfrac{1}{n}$:

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n}{\sqrt{n^2+1}} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{1+(1/n^2)}} = 1 > 0$. Since the harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges, so does $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n^2+1}}$.

22. Use the Limit Comparison Test with $a_n = \dfrac{2}{\sqrt{n}+2}$ and $b_n = \dfrac{1}{\sqrt{n}}$:

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{2\sqrt{n}}{\sqrt{n}+2} = \lim\limits_{n\to\infty} \dfrac{2}{1+2/\sqrt{n}} = 2 > 0$. Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ is a divergent p-series $[\,p = \frac{1}{2} \le 1]$, the series $\sum\limits_{n=1}^{\infty} \dfrac{2}{\sqrt{n}+2}$ is also divergent.

23. Use the Limit Comparison Test with $a_n = \dfrac{n+1}{n^3+n}$ and $b_n = \dfrac{1}{n^2}$:

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{(n+1)n^2}{n(n^2+1)} = \lim_{n\to\infty}\frac{n^2+n}{n^2+1} = \lim_{n\to\infty}\frac{1+1/n}{1+1/n^2} = 1 > 0$. Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series $[p = 2 > 1]$, the series $\sum_{n=1}^{\infty}\frac{n+1}{n^3+n}$ also converges.

24. Use the Limit Comparison Test with $a_n = \dfrac{n^2+n+1}{n^4+n^2}$ and $b_n = \dfrac{1}{n^2}$:

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{(n^2+n+1)n^2}{n^2(n^2+1)} = \lim_{n\to\infty}\frac{n^2+n+1}{n^2+1} = \lim_{n\to\infty}\frac{1+1/n+1/n^2}{1+1/n^2} = 1 > 0$. Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series $[p = 2 > 1]$, the series $\sum_{n=1}^{\infty}\frac{n^2+n+1}{n^4+n^2}$ also converges.

25. Use the Limit Comparison Test with $a_n = \dfrac{\sqrt{1+n}}{2+n}$ and $b_n = \dfrac{1}{\sqrt{n}}$:

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sqrt{1+n}\sqrt{n}}{2+n} = \lim_{n\to\infty}\frac{\sqrt{n^2+n}/\sqrt{n^2}}{(2+n)/n} = \lim_{n\to\infty}\frac{\sqrt{1+1/n}}{2/n+1} = 1 > 0$. Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a divergent p-series $[p = \frac{1}{2} \le 1]$, the series $\sum_{n=1}^{\infty}\frac{\sqrt{1+n}}{2+n}$ also diverges.

26. Use the Limit Comparison Test with $a_n = \dfrac{n+2}{(n+1)^3}$ and $b_n = \dfrac{1}{n^2}$:

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2(n+2)}{(n+1)^3} = \lim_{n\to\infty}\frac{1+\frac{2}{n}}{\left(1+\frac{1}{n}\right)^3} = 1 > 0$. Since $\sum_{n=3}^{\infty}\frac{1}{n^2}$ is a convergent (partial) p-series $[p = 2 > 1]$, the series $\sum_{n=3}^{\infty}\frac{n+2}{(n+1)^3}$ also converges.

27. Use the Limit Comparison Test with $a_n = \dfrac{5+2n}{(1+n^2)^2}$ and $b_n = \dfrac{1}{n^3}$:

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^3(5+2n)}{(1+n^2)^2} = \lim_{n\to\infty}\frac{5n^3+2n^4}{(1+n^2)^2}\cdot\frac{1/n^4}{1/(n^2)^2} = \lim_{n\to\infty}\frac{\frac{5}{n}+2}{\left(\frac{1}{n^2}+1\right)^2} = 2 > 0$. Since $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a convergent p-series $[p = 3 > 1]$, the series $\sum_{n=1}^{\infty}\frac{5+2n}{(1+n^2)^2}$ also converges.

28. $\dfrac{n+3^n}{n+2^n} > \dfrac{3^n}{n+2^n} > \dfrac{3^n}{2^n+2^n} = \dfrac{3^n}{2\cdot 2^n} = \dfrac{1}{2}\left(\dfrac{3}{2}\right)^n$, so the series $\displaystyle\sum_{n=1}^{\infty}\frac{n+3^n}{n+2^n}$ diverges by direct comparison with $\displaystyle\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$, which is a constant multiple of a divergent geometric series $[|r| = \frac{3}{2} > 1]$. *Or:* Use the Limit Comparison Test with $a_n = \dfrac{n+3^n}{n+2^n}$ and $b_n = \left(\dfrac{3}{2}\right)^n$.

29. $\dfrac{e^n+1}{ne^n+1} \ge \dfrac{e^n+1}{ne^n+n} = \dfrac{e^n+1}{n(e^n+1)} = \dfrac{1}{n}$ for $n \ge 1$, so the series $\displaystyle\sum_{n=1}^{\infty}\frac{e^n+1}{ne^n+1}$ diverges by direct comparison with the divergent harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$. *Or:* Use the Limit Comparison Test with $a_n = \dfrac{e^n+1}{ne^n+1}$ and $b_n = \dfrac{1}{n}$.