HW #5, Sec 11.6 Solutions

11.6 The Ratio and Root Tests

1. (a) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 8 > 1$, part (b) of the Ratio Test tells us that the series $\sum a_n$ is divergent.

(b) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0.8 < 1$, part (a) of the Ratio Test tells us that the series $\sum a_n$ is absolutely convergent (and therefore convergent).

(c) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test fails and the series $\sum a_n$ might converge or it might diverge.

2. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{a_n/a_{n+1}}\right| = \left|\frac{1}{2}\right| = \frac{1}{2} < 1$ Thus, the series $\sum a_n$ is absolutely convergent (and therefore convergent) by the Ratio Test.

3. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{5^{n+1}}\cdot\frac{5^n}{n}\right| = \lim_{n\to\infty}\left|\frac{1}{5}\cdot\frac{n+1}{n}\right| = \frac{1}{5}\lim_{n\to\infty}\frac{1+1/n}{1} = \frac{1}{5}(1) = \frac{1}{5} < 1$, so the series $\sum_{n=1}^{\infty}\frac{n}{5^n}$ is absolutely convergent by the Ratio Test.

4. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(-2)^n}\right| = \lim_{n\to\infty}\left|(-2)\frac{n^2}{(n+1)^2}\right| = 2\lim_{n\to\infty}\frac{1}{(1+1/n)^2} = 2(1) = 2 > 1$, so the series $\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}$ is divergent by the Ratio Test.

5. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n 3^{n+1}}{2^{n+1}(n+1)^3}\cdot\frac{2^n n^3}{(-1)^{n-1}3^n}\right| = \lim_{n\to\infty}\left|\left(-\frac{3}{2}\right)\frac{n^3}{(n+1)^3}\right| = \frac{3}{2}\lim_{n\to\infty}\frac{1}{(1+1/n)^3} = \frac{3}{2}(1) = \frac{3}{2} > 1$,

so the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n}{2^n n^3}$ is divergent by the Ratio Test.

6. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}}{[2(n+1)+1]!}\cdot\frac{(2n+1)!}{(-3)^n}\right| = \lim_{n\to\infty}\left|(-3)\frac{1}{(2n+3)(2n+2)}\right| = 3\lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)}$
$= 3(0) = 0 < 1$

so the series $\sum_{n=0}^{\infty}\frac{(-3)^n}{(2n+1)!}$ is absolutely convergent by the Ratio Test.

7. $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{1}{(k+1)!}\cdot\frac{k!}{1}\right| = \lim_{k\to\infty}\frac{1}{k+1} = 0 < 1$, so the series $\sum_{k=1}^{\infty}\frac{1}{k!}$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

8. $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)e^{-(k+1)}}{ke^{-k}}\right| = \lim_{k\to\infty}\left(\frac{k+1}{k}\cdot e^{-1}\right) = \frac{1}{e}\lim_{k\to\infty}\frac{1+1/k}{1} = \frac{1}{e}(1) = \frac{1}{e} < 1$, so the series $\sum_{k=1}^{\infty}ke^{-k}$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

9. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{10^{n+1}}{(n+2)\,4^{2n+3}}\cdot\frac{(n+1)\,4^{2n+1}}{10^n}\right] = \lim_{n\to\infty}\left(\frac{10}{4^2}\cdot\frac{n+1}{n+2}\right) = \frac{5}{8} < 1$, so the series $\sum_{n=1}^{\infty}\frac{10^n}{(n+1)4^{2n+1}}$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

10. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{(n+1)!}{100^{n+1}}\cdot\frac{100^n}{n!}\right] = \lim_{n\to\infty}\frac{n+1}{100} = \infty$, so the series $\sum_{n=1}^{\infty}\frac{n!}{100^n}$ diverges by the Ratio Test.

11. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\pi^{n+1}}{(-3)^n}\cdot\frac{(-3)^{n-1}}{n\pi^n}\right| = \lim_{n\to\infty}\left|\frac{\pi}{-3}\cdot\frac{n+1}{n}\right| = \frac{\pi}{3}\lim_{n\to\infty}\frac{1+1/n}{1} = \frac{\pi}{3}(1) = \frac{\pi}{3} > 1$, so the series $\sum_{n=1}^{\infty}\frac{n\pi^n}{(-3)^{n-1}}$ diverges by the Ratio Test. *Or:* Since $\lim_{n\to\infty}|a_n| = \infty$, the series diverges by the Test for Divergence.

12. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{10}}{(-10)^{n+2}}\cdot\frac{(-10)^{n+1}}{n^{10}}\right| = \lim_{n\to\infty}\left|\frac{1}{-10}\left(\frac{n+1}{n}\right)^{10}\right| = \frac{1}{10}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{10} = \frac{1}{10}(1) = \frac{1}{10} < 1$,

so the series $\sum_{n=1}^{\infty}\frac{n^{10}}{(-10)^{n+1}}$ is absolutely convergent by the Ratio Test.

13. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cos[(n+1)\pi/3]}{(n+1)!}\cdot\frac{n!}{\cos(n\pi/3)}\right| = \lim_{n\to\infty}\left|\frac{\cos[(n+1)\pi/3]}{(n+1)\cos(n\pi/3)}\right| = \lim_{n\to\infty}\frac{c}{n+1} = 0 < 1$ (where $0 < c \le 2$ for all positive integers n), so the series $\sum_{n=1}^{\infty}\frac{\cos(n\pi/3)}{n!}$ is absolutely convergent by the Ratio Test.

14. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right| = \lim_{n\to\infty}\frac{(n+1)n^n}{(n+1)^{n+1}} = \lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$, so the series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ is absolutely convergent by the Ratio Test.

15. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{100}100^{n+1}}{(n+1)!}\cdot\frac{n!}{n^{100}100^n}\right| = \lim_{n\to\infty}\frac{100}{n+1}\left(\frac{n+1}{n}\right)^{100} = \lim_{n\to\infty}\frac{100}{n+1}\left(1+\frac{1}{n}\right)^{100}$

$= 0\cdot 1 = 0 < 1$

so the series $\sum_{n=1}^{\infty}\frac{n^{100}100^n}{n!}$ is absolutely convergent by the Ratio Test.

16. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[2(n+1)]!}{[(n+1)!]^2}\cdot\frac{(n!)^2}{(2n)!}\right| = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \lim_{n\to\infty}\frac{(2+2/n)(2+1/n)}{(1+1/n)(1+1/n)} = \frac{2\cdot 2}{1\cdot 1} = 4 > 1$,

so the series $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$ diverges by the Ratio Test.

17. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n(n+1)!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)}\cdot\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(-1)^{n-1}n!}\right| = \lim_{n\to\infty}\frac{n+1}{2n+1}$

$= \lim_{n\to\infty}\frac{1+1/n}{2+1/n} = \frac{1}{2} < 1$,

so the series $1 - \frac{2!}{1\cdot 3} + \frac{3!}{1\cdot 3\cdot 5} - \frac{4!}{1\cdot 3\cdot 5\cdot 7} + \cdots + (-1)^{n-1}\frac{n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)} + \cdots$ is absolutely convergent by the Ratio Test.

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18. $\frac{2}{3}+\frac{2\cdot 5}{3\cdot 5}+\frac{2\cdot 5\cdot 8}{3\cdot 5\cdot 7}+\frac{2\cdot 5\cdot 8\cdot 11}{3\cdot 5\cdot 7\cdot 9}+\cdots=\sum\limits_{n=1}^{\infty}\frac{2\cdot 5\cdot 8\cdot 11\cdot\cdots\cdot(3n-1)}{3\cdot 5\cdot 7\cdot 9\cdot\cdots\cdot(2n+1)}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)(3n+2)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)(2n+3)}\cdot\frac{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}\right|$$
$$=\lim_{n\to\infty}\frac{3n+2}{2n+3}=\lim_{n\to\infty}\frac{3+2/n}{2+3/n}=\frac{3}{2}>1,$$

so the given series diverges by the Ratio Test.

19. $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left|\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)(2n+2)}{(n+1)!}\cdot\frac{n!}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}\right|=\lim\limits_{n\to\infty}\frac{2n+2}{n+1}=\lim\limits_{n\to\infty}\frac{2(n+1)}{n+1}=2>1$, so the series $\sum\limits_{n=1}^{\infty}\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{n!}$ diverges by the Ratio Test.

20. $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left|\frac{2^{n+1}(n+1)!}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)(3n+5)}\cdot\frac{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}{2^n n!}\right|=\lim\limits_{n\to\infty}\frac{2(n+1)}{3n+5}=\frac{2}{3}<1$, so the series $\sum\limits_{n=1}^{\infty}(-1)^n\frac{2^n n!}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}$ is absolutely convergent by the Ratio Test.

21. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\frac{n^2+1}{2n^2+1}=\lim\limits_{n\to\infty}\frac{1+1/n^2}{2+1/n^2}=\frac{1}{2}<1$, so the series $\sum\limits_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$ is absolutely convergent by the Root Test.

22. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\sqrt[n]{\left|\frac{(-2)^n}{n^n}\right|}=\lim\limits_{n\to\infty}\frac{2}{n}=0<1$, so the series $\sum\limits_{n=1}^{\infty}\frac{(-2)^n}{n^n}$ is absolutely convergent by the Root Test.

23. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^{n-1}}{(\ln n)^n}\right|}=\lim\limits_{n\to\infty}\frac{1}{\ln n}=0<1$, so the series $\sum\limits_{n=2}^{\infty}\frac{(-1)^{n-1}}{(\ln n)^n}$ is absolutely convergent by the Root Test.

24. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\sqrt[n]{\left|\left(\frac{-2n}{n+1}\right)^{5n}\right|}=\lim\limits_{n\to\infty}\frac{2^5n^5}{(n+1)^5}=32\lim\limits_{n\to\infty}\frac{1}{\left(\frac{n+1}{n}\right)^5}=32\lim\limits_{n\to\infty}\frac{1}{(1+1/n)^5}$

$$=32(1)=32>1,$$

so the series $\sum\limits_{n=1}^{\infty}\left(\frac{-2n}{n+1}\right)^{5n}$ diverges by the Root Test.

25. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}}=\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e>1$ [by Equation 3.6.6], so the series $\sum\limits_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$ diverges by the Root Test.

26. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\sqrt[n]{|(\arctan n)^n|}=\lim\limits_{n\to\infty}\arctan n=\frac{\pi}{2}>1$, so the series $\sum\limits_{n=0}^{\infty}(\arctan n)^n$ diverges by the Root Test.

27. $\sum\limits_{n=2}^{\infty}a_n=\sum\limits_{n=2}^{\infty}\frac{(-1)^n\ln n}{n}=\sum\limits_{n=2}^{\infty}(-1)^n b_n$. Now $b_n=\frac{\ln n}{n}>0$ for $n\geq 2$, and $\{b_n\}$ is decreasing for $n\geq 3$ since $\left(\frac{\ln x}{x}\right)'=\frac{x\cdot\frac{1}{x}-\ln x\cdot 1}{x^2}=\frac{1-\ln x}{x^2}<0$ when $\ln x>1$ or $x>e\approx 2.7$. Also, $\lim\limits_{n\to\infty}b_n=\lim\limits_{n\to\infty}\frac{\ln n}{n}\overset{H}{=}\lim\limits_{n\to\infty}\frac{1/n}{1}=0$, so the series $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ converges by the Alternating Series Test. To determine absolute convergence, note that

$\left|\frac{(-1)^n \ln n}{n}\right| = \frac{\ln n}{n} > \frac{1}{n}$ for $n \geq 3$, so $\sum_{n=2}^{\infty} \left|\frac{(-1)^n \ln n}{n}\right|$ is divergent by direct comparison with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is divergent.

Hence, $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ is conditionally convergent.

28. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{1-n}{2+3n}\right)^n\right|} = \lim_{n\to\infty} \frac{n-1}{3n+2} = \lim_{n\to\infty} \frac{1-1/n}{3+2/n} = \frac{1}{3} < 1$, so the series $\sum_{n=1}^{\infty} \left(\frac{1-n}{2+3n}\right)^n$ is absolutely convergent by the Root Test.

29. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-9)^{n+1}}{(n+1)10^{n+2}} \cdot \frac{n10^{n+1}}{(-9)^n}\right| = \lim_{n\to\infty} \left|\frac{(-9)n}{10(n+1)}\right| = \frac{9}{10} \lim_{n\to\infty} \frac{1}{1+1/n} = \frac{9}{10}(1) = \frac{9}{10} < 1$, so the series $\sum_{n=1}^{\infty} \frac{(-9)^n}{n10^{n+1}}$ is absolutely convergent by the Ratio Test.

30. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)5^{2n+2}}{10^{n+2}} \cdot \frac{10^{n+1}}{n5^{2n}}\right| = \lim_{n\to\infty} \frac{5^2(n+1)}{10n} = \frac{5}{2} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = \frac{5}{2}(1) = \frac{5}{2} > 1$, so the series $\sum_{n=1}^{\infty} \frac{n5^{2n}}{10^{n+1}}$ diverges by the Ratio Test. *Or:* Since $\lim_{n\to\infty} a_n = \infty$, the series diverges by the Test for Divergence.

31. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{\ln n}\right)^n\right|} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} \overset{H}{=} \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty$, so the series $\sum_{n=2}^{\infty} \left(\frac{n}{\ln n}\right)^n$ diverges by the Root Test.

32. $\left|\frac{\sin(n\pi/6)}{1+n\sqrt{n}}\right| \leq \frac{1}{1+n\sqrt{n}} < \frac{1}{n^{3/2}}$, so the series $\sum_{n=1}^{\infty} \frac{\sin(n\pi/6)}{1+n\sqrt{n}}$ converges by direct comparison with the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $(p = \frac{3}{2} > 1)$. It follows that the given series is absolutely convergent.

33. $\left|\frac{(-1)^n \arctan n}{n^2}\right| < \frac{\pi/2}{n^2}$, so since $\sum_{n=1}^{\infty} \frac{\pi/2}{n^2} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $(p = 2 > 1)$, the given series $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^2}$ converges absolutely by the Direct Comparison Test.

34. $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n} = \sum_{n=2}^{\infty} (-1)^n b_n$. Now $b_n = \frac{1}{\sqrt{n}\ln n} > 0$ for $n \geq 2$, $\{b_n\}$ is decreasing for $n \geq 2$, and $\lim_{n\to\infty} b_n = 0$, so the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n}$ converges by the Alternating Series Test. Also, observe that $\left|\frac{(-1)^n}{\sqrt{n}\ln n}\right| = \frac{1}{\sqrt{n}\ln n} > \frac{1}{\sqrt{n}\sqrt{n}} = \frac{1}{n}$ for $n \geq 2$, so the series $\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}\ln n}\right|$ is divergent by direct comparison with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent (partial) p-series $[p = 1 \leq 1]$. Thus, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n}$ is conditionally convergent.

35. By the recursive definition, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{5n+1}{4n+3}\right| = \frac{5}{4} > 1$, so the series diverges by the Ratio Test.

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36. By the recursive definition, $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\frac{2+\cos n}{\sqrt{n}}\right| = 0 < 1$, so the series converges absolutely by the Ratio Test.

37. The series $\sum\limits_{n=1}^{\infty}\frac{b_n^n \cos n\pi}{n} = \sum\limits_{n=1}^{\infty}(-1)^n\frac{b_n^n}{n}$, where $b_n > 0$ for $n \geq 1$ and $\lim\limits_{n\to\infty} b_n = \frac{1}{2}$.

$\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\frac{(-1)^{n+1}b_n^{n+1}}{n+1}\cdot\frac{n}{(-1)^n b_n^n}\right| = \lim\limits_{n\to\infty} b_n\frac{n}{n+1} = \frac{1}{2}(1) = \frac{1}{2} < 1$, so the series $\sum\limits_{n=1}^{\infty}\frac{b_n^n\cos n\pi}{n}$ is absolutely convergent by the Ratio Test.

38. $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)!}{(n+1)^{n+1}b_1b_2\cdots b_nb_{n+1}}\cdot\frac{n^nb_1b_2\cdots b_n}{(-1)^n\,n!}\right| = \lim\limits_{n\to\infty}\left|\frac{(-1)(n+1)n^n}{b_{n+1}(n+1)^{n+1}}\right| = \lim\limits_{n\to\infty}\frac{n^n}{b_{n+1}(n+1)^n}$

$$= \lim_{n\to\infty}\frac{1}{b_{n+1}}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\frac{1}{b_{n+1}}\left(\frac{1}{1+1/n}\right)^n = \lim_{n\to\infty}\frac{1}{b_{n+1}(1+1/n)^n} = \frac{1}{\frac{1}{2}e} = \frac{2}{e} < 1$$

so the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^n\,n!}{n^nb_1b_2b_3\cdots b_n}$ is absolutely convergent by the Ratio Test.

39. (a) $\lim\limits_{n\to\infty}\left|\frac{1/(n+1)^3}{1/n^3}\right| = \lim\limits_{n\to\infty}\frac{n^3}{(n+1)^3} = \lim\limits_{n\to\infty}\frac{1}{(1+1/n)^3} = 1$. Inconclusive for $\sum\limits_{n=1}^{\infty}\frac{1}{n^3}$.

(b) $\lim\limits_{n\to\infty}\left|\frac{(n+1)}{2^{n+1}}\cdot\frac{2^n}{n}\right| = \lim\limits_{n\to\infty}\frac{n+1}{2n} = \lim\limits_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2n}\right) = \frac{1}{2}$. Conclusive (convergent) for $\sum\limits_{n=1}^{\infty}\frac{n}{2^n}$.

(c) $\lim\limits_{n\to\infty}\left|\frac{(-3)^n}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-3)^{n-1}}\right| = 3\lim\limits_{n\to\infty}\sqrt{\frac{n}{n+1}} = 3\lim\limits_{n\to\infty}\sqrt{\frac{1}{1+1/n}} = 3$. Conclusive (divergent) for $\sum\limits_{n=1}^{\infty}\frac{(-3)^{n-1}}{\sqrt{n}}$.

(d) $\lim\limits_{n\to\infty}\left|\frac{\sqrt{n+1}}{1+(n+1)^2}\cdot\frac{1+n^2}{\sqrt{n}}\right| = \lim\limits_{n\to\infty}\left[\sqrt{1+\frac{1}{n}}\cdot\frac{1/n^2+1}{1/n^2+(1+1/n)^2}\right] = 1$. Inconclusive for $\sum\limits_{n=1}^{\infty}\frac{\sqrt{n}}{1+n^2}$.

40. We use the Ratio Test for the series $\sum\limits_{n=1}^{\infty}\frac{(n!)^2}{(kn)!}$:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)!]^2/[k(n+1)]!}{(n!)^2/(kn)!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{[k(n+1)]\,[k(n+1)-1]\cdots[kn+1]}\right|$$

Now if $k = 1$, then this is equal to $\lim\limits_{n\to\infty}\left|\frac{(n+1)^2}{(n+1)}\right| = \infty$, so the series diverges; if $k = 2$, the limit is

$\lim\limits_{n\to\infty}\left|\frac{(n+1)^2}{(2n+2)(2n+1)}\right| = \frac{1}{4} < 1$, so the series converges, and if $k > 2$, then the highest power of n in the denominator is larger than 2, and so the limit is 0, indicating convergence. So the series converges for $k \geq 2$.

41. (a) $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim\limits_{n\to\infty}\left|\frac{x}{n+1}\right| = |x|\lim\limits_{n\to\infty}\frac{1}{n+1} = |x|\cdot 0 = 0 < 1$, so by the Ratio Test the series $\sum\limits_{n=0}^{\infty}\frac{x^n}{n!}$ converges for all x.

(b) Since the series of part (a) always converges, we must have $\lim\limits_{n\to\infty}\frac{x^n}{n!} = 0$ by Theorem 11.2.6.

42. (a) $R_n = a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4} + \cdots = a_{n+1}\left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+1}} + \frac{a_{n+4}}{a_{n+1}} + \cdots\right)$

$$= a_{n+1}\left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+2}}\frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+4}}{a_{n+3}}\frac{a_{n+3}}{a_{n+2}}\frac{a_{n+2}}{a_{n+1}} + \cdots\right)$$

$$= a_{n+1}(1 + r_{n+1} + r_{n+2}r_{n+1} + r_{n+3}r_{n+2}r_{n+1} + \cdots) \quad (\star)$$

$$\leq a_{n+1}\left(1 + r_{n+1} + r_{n+1}^2 + r_{n+1}^3 + \cdots\right) \quad \text{[since } \{r_n\} \text{ is decreasing]} \quad = \frac{a_{n+1}}{1 - r_{n+1}}$$

(b) Note that since $\{r_n\}$ is increasing and $r_n \to L$ as $n \to \infty$, we have $r_n < L$ for all n. So, starting with equation ($\star$),

$$R_n = a_{n+1}(1 + r_{n+1} + r_{n+2}r_{n+1} + r_{n+3}r_{n+2}r_{n+1} + \cdots) \leq a_{n+1}\left(1 + L + L^2 + L^3 + \cdots\right) = \frac{a_{n+1}}{1 - L}.$$

43. (a) $s_5 = \sum_{n=1}^{5} \frac{1}{n2^n} = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \frac{1}{160} = \frac{661}{960} \approx 0.68854$. Now the ratios

$r_n = \frac{a_{n+1}}{a_n} = \frac{n2^n}{(n+1)2^{n+1}} = \frac{n}{2(n+1)}$ form an increasing sequence, since

$r_{n+1} - r_n = \frac{n+1}{2(n+2)} - \frac{n}{2(n+1)} = \frac{(n+1)^2 - n(n+2)}{2(n+1)(n+2)} = \frac{1}{2(n+1)(n+2)} > 0$. So by Exercise 42(b), the error

in using s_5 is $R_5 \leq \frac{a_6}{1 - \lim\limits_{n\to\infty} r_n} = \frac{1/(6 \cdot 2^6)}{1 - 1/2} = \frac{1}{192} \approx 0.00521$.

(b) The error in using s_n as an approximation to the sum is $R_n = \frac{a_{n+1}}{1 - \frac{1}{2}} = \frac{2}{(n+1)2^{n+1}}$. We want $R_n < 0.00005 \Leftrightarrow$

$\frac{1}{(n+1)2^n} < 0.00005 \Leftrightarrow (n+1)2^n > 20{,}000$. To find such an n we can use trial and error or a graph. We calculate

$(11+1)2^{11} = 24{,}576$, so $s_{11} = \sum_{n=1}^{11} \frac{1}{n2^n} \approx 0.693109$ is within 0.00005 of the actual sum.

44. $s_{10} = \sum_{n=1}^{10} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{10}{1024} \approx 1.988$. The ratios $r_n = \frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} = \frac{1}{2}\left(1 + \frac{1}{n}\right)$ form a decreasing sequence, and $r_{11} = \frac{11+1}{2(11)} = \frac{12}{22} = \frac{6}{11} < 1$, so by Exercise 42(a), the error in using s_{10} to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ is $R_{10} \leq \frac{a_{11}}{1 - r_{11}} = \frac{\frac{11}{2048}}{1 - \frac{6}{11}} = \frac{121}{10{,}240} \approx 0.0118$.

45. (i) Following the hint, we get that $|a_n| < r^n$ for $n \geq N$, and so since the geometric series $\sum_{n=1}^{\infty} r^n$ converges $[0 < r < 1]$, the series $\sum_{n=N}^{\infty} |a_n|$ converges as well by the Direct Comparison Test, and hence so does $\sum_{n=1}^{\infty} |a_n|$, so $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then there is an integer N such that $\sqrt[n]{|a_n|} > 1$ for all $n \geq N$, so $|a_n| > 1$ for $n \geq N$. Thus, $\lim\limits_{n\to\infty} a_n \neq 0$, so $\sum_{n=1}^{\infty} a_n$ diverges by the Test for Divergence.

(iii) Consider $\sum_{n=1}^{\infty} \frac{1}{n}$ [diverges] and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ [converges]. For each sum, $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, so the Root Test is inconclusive.