HW #7, PART II SOLUTIONS SEC. 10.2

10.2 Calculus with Parametric Curves

1. $x = 2t^3 + 3t,\ y = 4t - 5t^2 \;\Rightarrow\; \dfrac{dx}{dt} = 6t^2 + 3,\ \dfrac{dy}{dt} = 4 - 10t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4 - 10t}{6t^2 + 3}$.

2. $x = t - \ln t,\ y = t^2 - t^{-2} \;\Rightarrow\; \dfrac{dx}{dt} = 1 - t^{-1},\ \dfrac{dy}{dt} = 2t + 2t^{-3}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t + 2t^{-3}}{1 - t^{-1}} \cdot \dfrac{t^3}{t^3} = \dfrac{2t^4 + 2}{t^3 - t^2}$.

3. $x = te^t,\ y = t + \sin t \;\Rightarrow\; \dfrac{dx}{dt} = te^t + e^t = e^t(t+1),\ \dfrac{dy}{dt} = 1 + \cos t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 + \cos t}{e^t(t+1)}$.

4. $x = t + \sin(t^2 + 2),\ y = \tan(t^2 + 2) \;\Rightarrow\; \dfrac{dx}{dt} = 1 + 2t\cos(t^2 + 2),\ \dfrac{dy}{dt} = 2t\sec^2(t^2 + 2)$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t\sec^2(t^2 + 2)}{1 + 2t\cos(t^2 + 2)}$.

5. $x = t^2 + 2t,\ y = 2^t - 2t;\ (15, 2)$. $\dfrac{dy}{dt} = 2^t \ln 2 - 2,\ \dfrac{dx}{dt} = 2t + 2$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2^t \ln 2 - 2}{2t + 2}$.

At $(15, 2)$, $x = t^2 + 2t = 15 \;\Rightarrow\; t^2 + 2t - 15 = 0 \;\Rightarrow\; (t+5)(t-3) = 0 \;\Rightarrow\; t = -5$ or $t = 3$. Only $t = 3$ gives $y = 2$. With $t = 3$, $\dfrac{dy}{dx} = \dfrac{2^3 \ln 2 - 2}{2(3) + 2} = \dfrac{4\ln 2 - 1}{4} = \ln 2 - \dfrac{1}{4} \approx 0.44$.

6. $x = t + \cos \pi t,\ y = -t + \sin \pi t;\ (3, -2)$. $\dfrac{dy}{dt} = -1 + \pi \cos \pi t,\ \dfrac{dx}{dt} = 1 - \pi \sin \pi t$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-1 + \pi\cos \pi t}{1 - \pi \sin \pi t}$. When $x = 3$, we have $t + \cos \pi t = 3 \;\Rightarrow\; \cos^2 \pi t = (3 - t)^2$ **(1)**. When $y = -2$, we have $-t + \sin \pi t = -2 \;\Rightarrow\; \sin^2 \pi t = (t - 2)^2$ **(2)**. Adding **(1)** and **(2)** gives $\sin^2 \pi t + \cos^2 \pi t = (t-2)^2 + (3-t)^2 \;\Rightarrow\; 1 = t^2 - 4t + 4 + 9 - 6t + t^2 \;\Rightarrow\; 0 = 2t^2 - 10t + 12 \;\Rightarrow\; 0 = 2(t-2)(t-3) \;\Rightarrow\; t = 2$ or $t = 3$. Only $t = 2$ gives $y = -2$. With $t = 2$, $\dfrac{dy}{dx} = \dfrac{-1 + \pi \cos 2\pi}{1 - \pi \sin 2\pi} = \dfrac{-1 + \pi}{1 - 0} = \pi - 1 \approx 2.14$.

7. $x = t^3 + 1,\ y = t^4 + t;\ \ t = -1$. $\dfrac{dy}{dt} = 4t^3 + 1,\ \dfrac{dx}{dt} = 3t^2$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4t^3 + 1}{3t^2}$. When $t = -1$, $(x, y) = (0, 0)$ and $dy/dx = -3/3 = -1$, so an equation of the tangent to the curve at the point corresponding to $t = -1$ is $y - 0 = -1(x - 0)$, or $y = -x$.

8. $x = \sqrt{t},\ y = t^2 - 2t;\ \ t = 4$. $\dfrac{dy}{dt} = 2t - 2,\ \dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = (2t - 2)2\sqrt{t} = 4(t-1)\sqrt{t}$. When $t = 4$, $(x, y) = (2, 8)$ and $dy/dx = 4(3)(2) = 24$, so an equation of the tangent to the curve at the point corresponding to $t = 4$ is $y - 8 = 24(x - 2)$, or $y = 24x - 40$.

9. $x = \sin 2t + \cos t,\ y = \cos 2t - \sin t;\ t = \pi$. $\dfrac{dy}{dt} = -2\sin 2t - \cos t,\ \dfrac{dx}{dt} = 2\cos 2t - \sin t$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-2\sin 2t - \cos t}{2\cos 2t - \sin t}$. When $t = \pi$, $(x, y) = (-1, 1)$, and $\dfrac{dy}{dx} = \dfrac{1}{2}$, so an equation of the tangent to the curve at the point corresponding to $t = \pi$ is $y - 1 = \frac{1}{2}[x - (-1)]$, or $y = \frac{1}{2}x + \frac{3}{2}$.

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10. $x = e^t \sin \pi t,\ y = e^{2t};\quad t = 0.\ \dfrac{dy}{dt} = 2e^{2t},\ \dfrac{dx}{dt} = e^t(\pi \cos \pi t) + (\sin \pi t)e^t = e^t(\pi \cos \pi t + \sin \pi t)$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2e^{2t}}{e^t(\pi \cos \pi t + \sin \pi t)} = \dfrac{2e^t}{\pi \cos \pi t + \sin \pi t}$. When $t = 0$, $(x, y) = (0, 1)$ and $\dfrac{dy}{dx} = \dfrac{2}{\pi}$, so an equation of the tangent to the curve at the point corresponding to $t = 0$ is $y - 1 = \frac{2}{\pi}(x - 0)$, or $y = \frac{2}{\pi}x + 1$.

11. (a) $x = \sin t,\ y = \cos^2 t;\ \left(\frac{1}{2}, \frac{3}{4}\right).\ \dfrac{dy}{dt} = 2\cos t(-\sin t),\ \dfrac{dx}{dt} = \cos t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-2\sin t \cos t}{\cos t} = -2\sin t$.

At $\left(\frac{1}{2}, \frac{3}{4}\right)$, $x = \sin t = \frac{1}{2} \Rightarrow t = \frac{\pi}{6}$, so $dy/dx = -2\sin\frac{\pi}{6} = -2\left(\frac{1}{2}\right) = -1$, and an equation of the tangent is $y - \frac{3}{4} = -1\left(x - \frac{1}{2}\right)$, or $y = -x + \frac{5}{4}$.

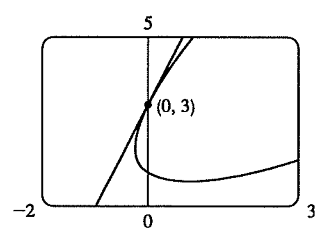
(b) $x = \sin t \Rightarrow x^2 = \sin^2 t = 1 - \cos^2 t = 1 - y$, so $y = 1 - x^2$, and $y' = -2x$. At $\left(\frac{1}{2}, \frac{3}{4}\right)$, $y' = -2 \cdot \frac{1}{2} = -1$, so an equation of the tangent is $y - \frac{3}{4} = -1\left(x - \frac{1}{2}\right)$, or $y = -x + \frac{5}{4}$.

12. (a) $x = \sqrt{t+4},\ y = 1/(t+4);\ \left(2, \frac{1}{4}\right).\ \dfrac{dy}{dt} = -\dfrac{1}{(t+4)^2},\ \dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t+4}}$, and

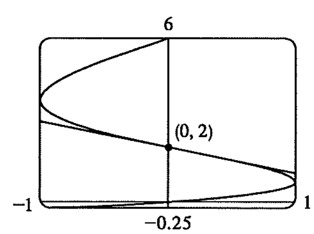
$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-1/(t+4)^2}{1/\left(2\sqrt{t+4}\right)} = -2(t+4)^{-3/2}$. At $\left(2, \frac{1}{4}\right)$, $x = \sqrt{t+4} = 2 \Rightarrow t + 4 = 4 \Rightarrow t = 0$ and $dy/dx = -2(4)^{-3/2} = -\frac{1}{4}$, so an equation of the tangent is $y - \frac{1}{4} = -\frac{1}{4}(x - 2)$, or $y = -\frac{1}{4}x + \frac{3}{4}$.

(b) $x = \sqrt{t+4} \Rightarrow x^2 = t + 4 \Rightarrow t = x^2 - 4$, so $y = \dfrac{1}{t+4} = \dfrac{1}{x^2 - 4 + 4} = \dfrac{1}{x^2}$, and $y' = -\dfrac{2}{x^3}$. At $\left(2, \frac{1}{4}\right)$, $y' = -2/2^3 = -1/4$, so an equation of the tangent is $y - \frac{1}{4} = -\frac{1}{4}(x - 2)$, or $y = -\frac{1}{4}x + \frac{3}{4}$.

13. $x = t^2 - t,\ y = t^2 + t + 1;\ (0, 3).\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{2t-1}$. To find the value of t corresponding to the point $(0, 3)$, solve $x = 0 \Rightarrow t^2 - t = 0 \Rightarrow t(t - 1) = 0 \Rightarrow t = 0$ or $t = 1$. Only $t = 1$ gives $y = 3$. With $t = 1$, $dy/dx = 3$, and an equation of the tangent is $y - 3 = 3(x - 0)$, or $y = 3x + 3$.



14. $x = \sin \pi t,\ y = t^2 + t;\ (0, 2).\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{\pi \cos \pi t}$. To find the value of t corresponding to the point $(0, 2)$, solve $y = 2 \Rightarrow t^2 + t - 2 = 0 \Rightarrow (t + 2)(t - 1) = 0 \Rightarrow t = -2$ or $t = 1$. Either value gives $dy/dx = -3/\pi$, so an equation of the tangent is $y - 2 = -\frac{3}{\pi}(x - 0)$, or $y = -\frac{3}{\pi}x + 2$.



15. $x = t^2 + 1,\ y = t^2 + t \Rightarrow \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{2t} = 1 + \dfrac{1}{2t} \Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{-1/(2t^2)}{2t} = -\dfrac{1}{4t^3}$.

The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $t < 0$.

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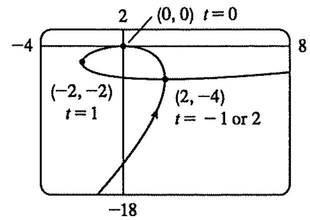
22. $x = t^3 - 3t,\ y = t^3 - 3t^2$. $\dfrac{dy}{dt} = 3t^2 - 6t = 3t(t-2)$, so $\dfrac{dy}{dt} = 0 \;\Leftrightarrow$ $t = 0$ or $2 \;\Leftrightarrow\; (x, y) = (0, 0)$ or $(2, -4)$. $\dfrac{dx}{dt} = 3t^2 - 3 = 3(t+1)(t-1)$, so $\dfrac{dx}{dt} = 0 \;\Leftrightarrow\; t = -1$ or $1 \;\Leftrightarrow\; (x, y) = (2, -4)$ or $(-2, -2)$. The curve has horizontal tangents at $(0, 0)$ and $(2, -4)$, and vertical tangents at $(2, -4)$ and $(-2, -2)$.



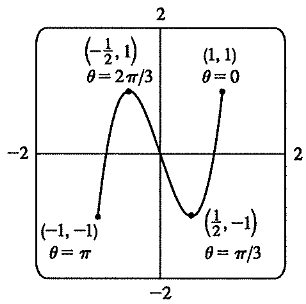
23. $x = \cos\theta,\ y = \cos 3\theta$. The whole curve is traced out for $0 \le \theta \le \pi$.

$\dfrac{dy}{d\theta} = -3\sin 3\theta$, so $\dfrac{dy}{d\theta} = 0 \;\Leftrightarrow\; \sin 3\theta = 0 \;\Leftrightarrow\; 3\theta = 0, \pi, 2\pi$, or $3\pi \;\Leftrightarrow$

$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$, or $\pi \;\Leftrightarrow\; (x, y) = (1, 1), \left(\frac{1}{2}, -1\right), \left(-\frac{1}{2}, 1\right)$, or $(-1, -1)$.

$\dfrac{dx}{d\theta} = -\sin\theta$, so $\dfrac{dx}{d\theta} = 0 \;\Leftrightarrow\; \sin\theta = 0 \;\Leftrightarrow\; \theta = 0$ or $\pi \;\Leftrightarrow$

$(x, y) = (1, 1)$ or $(-1, -1)$. Both $\dfrac{dy}{d\theta}$ and $\dfrac{dx}{d\theta}$ equal 0 when $\theta = 0$ and π.



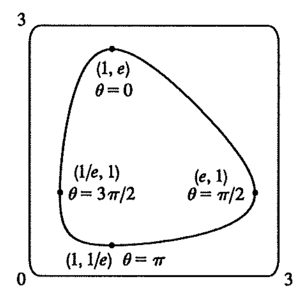
To find the slope when $\theta = 0$, we find $\displaystyle\lim_{\theta\to 0}\frac{dy}{dx} = \lim_{\theta\to 0}\frac{-3\sin 3\theta}{-\sin\theta} \overset{\text{H}}{=} \lim_{\theta\to 0}\frac{-9\cos 3\theta}{-\cos\theta} = 9$, which is the same slope when $\theta = \pi$. Thus, the curve has horizontal tangents at $\left(\frac{1}{2}, -1\right)$ and $\left(-\frac{1}{2}, 1\right)$, and there are no vertical tangents.

24. $x = e^{\sin\theta},\ y = e^{\cos\theta}$. The whole curve is traced out for $0 \le \theta < 2\pi$.

$\dfrac{dy}{d\theta} = -\sin\theta\, e^{\cos\theta}$, so $\dfrac{dy}{d\theta} = \theta \;\Leftrightarrow\; \sin\theta = 0 \;\Leftrightarrow\; \theta = 0$ or $\pi \;\Leftrightarrow$

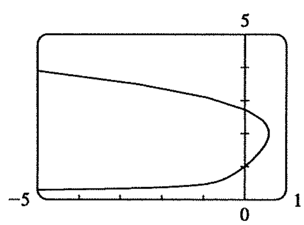
$(x, y) = (1, e)$ or $(1, 1/e)$. $\dfrac{dx}{d\theta} = \cos\theta\, e^{\sin\theta}$, so $\dfrac{dx}{d\theta} = 0 \;\Leftrightarrow\; \cos\theta = 0 \;\Leftrightarrow$

$\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2} \;\Leftrightarrow\; (x, y) = (e, 1)$ or $(1/e, 1)$. The curve has horizontal tangents at $(1, e)$ and $(1, 1/e)$, and vertical tangents at $(e, 1)$ and $(1/e, 1)$.

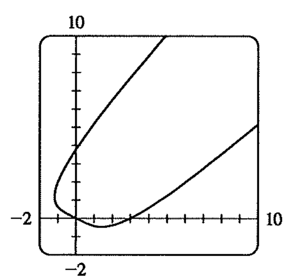


25. From the graph, it appears that the rightmost point on the curve $x = t - t^6$, $y = e^t$ is about $(0.6, 2)$. To find the exact coordinates, we find the value of t for which the graph has a vertical tangent, that is, $0 = dx/dt = 1 - 6t^5 \;\Leftrightarrow\; t = 1/\sqrt[5]{6}$. Hence, the rightmost point is

$\left(1/\sqrt[5]{6} - 1/\left(6\sqrt[5]{6}\right), e^{1/\sqrt[5]{6}}\right) = \left(5 \cdot 6^{-6/5}, e^{6^{-1/5}}\right) \approx (0.58, 2.01)$.



26. From the graph, it appears that the lowest point and the leftmost point on the curve $x = t^4 - 2t$, $y = t + t^4$ are $(1.5, -0.5)$ and $(-1.2, 1.2)$, respectively. To find the exact coordinates, we solve $dy/dt = 0$ (horizontal tangents) and $dx/dt = 0$ (vertical tangents).



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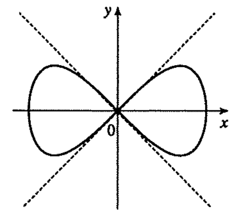
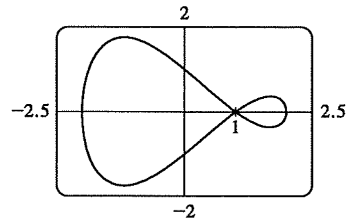


29. $x = \cos t,\ y = \sin t \cos t.\ \ dx/dt = -\sin t,$

$dy/dt = -\sin^2 t + \cos^2 t = \cos 2t.$ $(x, y) = (0, 0) \iff \cos t = 0 \iff t$ is an odd multiple of $\frac{\pi}{2}$. When $t = \frac{\pi}{2}$, $dx/dt = -1$ and $dy/dt = -1$, so $dy/dx = 1$. When $t = \frac{3\pi}{2}$, $dx/dt = 1$ and $dy/dt = -1$. So $dy/dx = -1$. Thus, $y = x$ and $y = -x$ are both tangent to the curve at $(0, 0)$.



30. $x = -2\cos t,\ y = \sin t + \sin 2t$. From the graph, it appears that the curve crosses itself at the point $(1, 0)$. If this is true, then $x = 1 \iff$
$-2\cos t = 1 \iff \cos t = -\frac{1}{2} \iff t = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$ for $0 \le t \le 2\pi$.
Substituting either value of t into y gives $y = 0$, confirming that $(1, 0)$ is the point where the curve crosses itself. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\cos t + 2\cos 2t}{2\sin t}$.

When $t = \dfrac{2\pi}{3}$, $\dfrac{dy}{dx} = \dfrac{-1/2 + 2(-1/2)}{2(\sqrt{3}/2)} = \dfrac{-3/2}{\sqrt{3}} = -\dfrac{\sqrt{3}}{2}$, so an equation of the tangent line is $y - 0 = -\dfrac{\sqrt{3}}{2}(x - 1)$,

or $y = -\dfrac{\sqrt{3}}{2}x + \dfrac{\sqrt{3}}{2}$. Similarly, when $t = \dfrac{4\pi}{3}$, an equation of the tangent line is $y = \dfrac{\sqrt{3}}{2}x - \dfrac{\sqrt{3}}{2}$.

31. $x = r\theta - d\sin\theta,\ y = r - d\cos\theta$.

(a) $\dfrac{dx}{d\theta} = r - d\cos\theta$, $\dfrac{dy}{d\theta} = d\sin\theta$, so $\dfrac{dy}{dx} = \dfrac{d\sin\theta}{r - d\cos\theta}$.

(b) If $0 < d < r$, then $|d\cos\theta| \le d < r$, so $r - d\cos\theta \ge r - d > 0$. This shows that $dx/d\theta$ never vanishes, so the trochoid can have no vertical tangent if $d < r$.

32. $x = a\cos^3\theta,\ y = a\sin^3\theta$.

(a) $\dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$, $\dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$, so $\dfrac{dy}{dx} = -\dfrac{\sin\theta}{\cos\theta} = -\tan\theta$.

(b) The tangent is horizontal $\iff dy/dx = 0 \iff \tan\theta = 0 \iff \theta = n\pi \iff (x, y) = (\pm a, 0)$.

The tangent is vertical $\iff \cos\theta = 0 \iff \theta$ is an odd multiple of $\frac{\pi}{2} \iff (x, y) = (0, \pm a)$.

(c) $dy/dx = \pm 1 \iff \tan\theta = \pm 1 \iff \theta$ is an odd multiple of $\frac{\pi}{4} \iff (x, y) = \left(\pm\frac{\sqrt{2}}{4}a, \pm\frac{\sqrt{2}}{4}a\right)$

[All sign choices are valid.]

33. $x = 3t^2 + 1,\ y = t^3 - 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2}{6t} = \dfrac{t}{2}$. The tangent line has slope $\dfrac{1}{2}$ when $\dfrac{t}{2} = \dfrac{1}{2} \iff t = 1$, so the point is $(4, 0)$.

34. $x = 3t^2 + 1, y = 2t^3 + 1, \dfrac{dx}{dt} = 6t, \dfrac{dy}{dt} = 6t^2$, so $\dfrac{dy}{dx} = \dfrac{6t^2}{6t} = t$ [even where $t = 0$].

So at the point corresponding to parameter value t, an equation of the tangent line is $y - (2t^3 + 1) = t[x - (3t^2 + 1)]$.

If this line is to pass through $(4, 3)$, we must have $3 - (2t^3 + 1) = t[4 - (3t^2 + 1)] \iff 2t^3 - 2 = 3t^3 - 3t \iff$

45. $x = t - 2\sin t,\ y = 1 - 2\cos t,\ 0 \le t \le 4\pi.$ $dx/dt = 1 - 2\cos t$ and $dy/dt = 2\sin t$, so

$(dx/dt)^2 + (dy/dt)^2 = (1 - 2\cos t)^2 + (2\sin t)^2 = 1 - 4\cos t + 4\cos^2 t + 4\sin^2 t = 5 - 4\cos t$. Thus,

$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_0^{4\pi} \sqrt{5 - 4\cos t}\, dt \approx 26.7298.$

46. $x = t\cos t,\ y = t - 5\sin t.$ $dx/dt = \cos t - t\sin t$ and $dy/dt = 1 - 5\cos t$, so

$(dx/dt)^2 + (dy/dt)^2 = (\cos t - t\sin t)^2 + (1 - 5\cos t)^2$. Observe that when $t = -\pi$, $(x, y) = (\pi, -\pi)$ and when $t = \pi$,

$(x, y) = (-\pi, \pi)$. Thus, $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_{-\pi}^{\pi} \sqrt{(\cos t - t\sin t)^2 + (1 - 5\cos t)^2}\, dt \approx 22.8546.$

47. $x = \frac{2}{3}t^3,\ y = t^2 - 2,\ 0 \le t \le 3.$ $dx/dt = 2t^2$ and $dy/dt = 2t$, so $(dx/dt)^2 + (dy/dt)^2 = 4t^4 + 4t^2 = 4t^2(t^2 + 1)$.

Thus,

$$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_0^3 \sqrt{4t^2(t^2+1)}\, dt = \int_0^3 2t\sqrt{t^2+1}\, dt$$

$$= \int_1^{10} \sqrt{u}\, du \quad [u = t^2 + 1,\ du = 2t\, dt] \quad = \left[\tfrac{2}{3}u^{3/2}\right]_1^{10} = \tfrac{2}{3}(10^{3/2} - 1) = \tfrac{2}{3}\left(10\sqrt{10} - 1\right)$$

48. $x = e^t - t,\ y = 4e^{t/2},\ 0 \le t \le 2.$ $dx/dt = e^t - 1$ and $dy/dt = 2e^{t/2}$, so

$(dx/dt)^2 + (dy/dt)^2 = (e^t - 1)^2 + (2e^{t/2})^2 = e^{2t} - 2e^t + 1 + 4e^t = e^{2t} + 2e^t + 1 = (e^t + 1)^2$. Thus,

$$L = \int_0^2 \sqrt{(e^t+1)^2}\, dt = \int_0^2 \left|e^t + 1\right|\, dt = \int_0^2 (e^t + 1)\, dt = \left[e^t + t\right]_0^2 = (e^2 + 2) - (1 + 0) = e^2 + 1.$$

49. $x = t\sin t,\ y = t\cos t,\ 0 \le t \le 1.$ $\dfrac{dx}{dt} = t\cos t + \sin t$ and $\dfrac{dy}{dt} = -t\sin t + \cos t$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = t^2\cos^2 t + 2t\sin t\,\cos t + \sin^2 t + t^2\sin^2 t - 2t\sin t\,\cos t + \cos^2 t$$

$$= t^2(\cos^2 t + \sin^2 t) + \sin^2 t + \cos^2 t = t^2 + 1.$$

Thus, $L = \int_0^1 \sqrt{t^2+1}\, dt \overset{21}{=} \left[\frac{1}{2}t\sqrt{t^2+1} + \frac{1}{2}\ln\!\left(t + \sqrt{t^2+1}\right)\right]_0^1 = \frac{1}{2}\sqrt{2} + \frac{1}{2}\ln\!\left(1 + \sqrt{2}\right).$

50. $x = 3\cos t - \cos 3t,\ y = 3\sin t - \sin 3t,\ 0 \le t \le \pi.$ $\dfrac{dx}{dt} = -3\sin t + 3\sin 3t$ and $\dfrac{dy}{dt} = 3\cos t - 3\cos 3t$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\sin^2 t - 18\sin t\,\sin 3t + 9\sin^2 3t + 9\cos^2 t - 18\cos t\,\cos 3t + 9\cos^2 3t$$

$$= 9(\cos^2 t + \sin^2 t) - 18(\cos t\,\cos 3t + \sin t\,\sin 3t) + 9(\cos^2 3t + \sin^2 3t)$$

$$= 9(1) - 18\cos(t - 3t) + 9(1) = 18 - 18\cos(-2t) = 18(1 - \cos 2t)$$

$$= 18[1 - (1 - 2\sin^2 t)] = 36\sin^2 t.$$

Thus, $L = \int_0^\pi \sqrt{36\sin^2 t}\, dt = 6\int_0^\pi |\sin t|\, dt = 6\int_0^\pi \sin t\, dt = -6\left[\cos t\right]_0^\pi = -6\,(-1 - 1) = 12.$