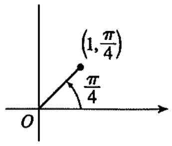
HW 7, PART III SOLUTIONS SEC 10.3

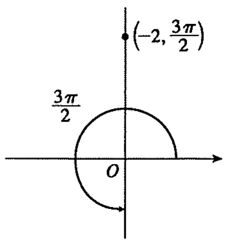
10.3 Polar Coordinates

1. (a) $\left(1, \frac{\pi}{4}\right)$



By adding 2π to $\frac{\pi}{4}$, we obtain the point $\left(1, \frac{9\pi}{4}\right)$, which satisfies the $r > 0$ requirement. The direction opposite $\frac{\pi}{4}$ is $\frac{5\pi}{4}$, so $\left(-1, \frac{5\pi}{4}\right)$ is a point that satisfies the $r < 0$ requirement.

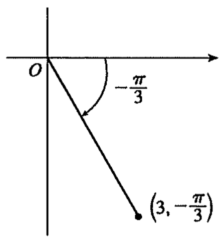
(b) $\left(-2, \frac{3\pi}{2}\right)$



$r > 0$: $\left(-(-2), \frac{3\pi}{2} - \pi\right) = \left(2, \frac{\pi}{2}\right)$

$r < 0$: $\left(-2, \frac{3\pi}{2} + 2\pi\right) = \left(-2, \frac{7\pi}{2}\right)$

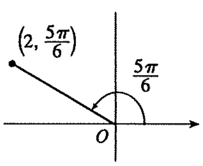
(c) $\left(3, -\frac{\pi}{3}\right)$



$r > 0$: $\left(3, -\frac{\pi}{3} + 2\pi\right) = \left(3, \frac{5\pi}{3}\right)$

$r < 0$: $\left(-3, -\frac{\pi}{3} + \pi\right) = \left(-3, \frac{2\pi}{3}\right)$

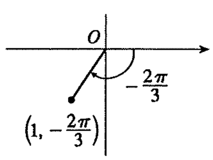
2. (a) $\left(2, \frac{5\pi}{6}\right)$



$r > 0$: $\left(2, \frac{5\pi}{6} + 2\pi\right) = \left(2, \frac{17\pi}{6}\right)$

$r < 0$: $\left(-2, \frac{5\pi}{6} - \pi\right) = \left(-2, -\frac{\pi}{6}\right)$

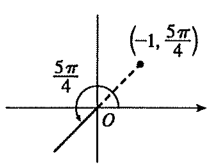
(b) $\left(1, -\frac{2\pi}{3}\right)$



$r > 0$: $\left(1, -\frac{2\pi}{3} + 2\pi\right) = \left(1, \frac{4\pi}{3}\right)$

$r < 0$: $\left(-1, -\frac{2\pi}{3} + \pi\right) = \left(-1, \frac{\pi}{3}\right)$

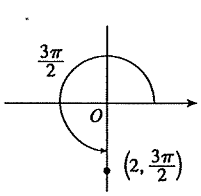
(c) $\left(-1, \frac{5\pi}{4}\right)$



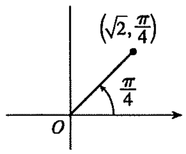
$r > 0$: $\left(-(-1), \frac{5\pi}{4} - \pi\right) = \left(1, \frac{\pi}{4}\right)$

$r < 0$: $\left(-1, \frac{5\pi}{4} - 2\pi\right) = \left(-1, -\frac{3\pi}{4}\right)$

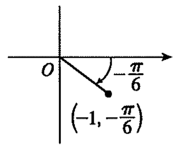
3. (a)



$x = 2\cos\frac{3\pi}{2} = 2(0) = 0$ and $y = 2\sin\frac{3\pi}{2} = 2(-1) = -2$ give us the Cartesian coordinates $(0, -2)$.

(b) 

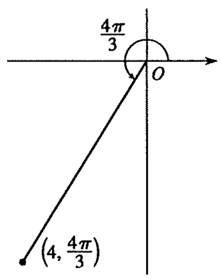
$x = \sqrt{2}\cos\frac{\pi}{4} = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1$ and $y = \sqrt{2}\sin\frac{\pi}{4} = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1$ give us the Cartesian coordinates $(1, 1)$.

(c) 

$x = -1\cos\left(-\frac{\pi}{6}\right) = -1\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}$ and

$y = -1\sin\left(-\frac{\pi}{6}\right) = -1\left(-\frac{1}{2}\right) = \frac{1}{2}$ give us the Cartesian

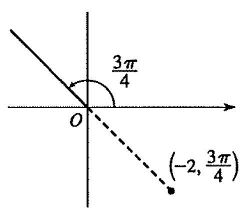
coordinates $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

4. (a) 

$x = 4\cos\frac{4\pi}{3} = 4\left(-\frac{1}{2}\right) = -2$ and

$y = 4\sin\frac{4\pi}{3} = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$ give us the Cartesian

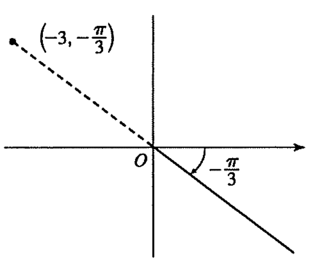
coordinates $\left(-2, -2\sqrt{3}\right)$.

(b) 

$x = -2\cos\frac{3\pi}{4} = -2\left(-\frac{\sqrt{2}}{2}\right) = \sqrt{2}$ and

$y = -2\sin\frac{3\pi}{4} = -2\left(\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$ give us the Cartesian

coordinates $\left(\sqrt{2}, -\sqrt{2}\right)$.

(c) 

$x = -3\cos\left(-\frac{\pi}{3}\right) = -3\left(\frac{1}{2}\right) = -\frac{3}{2}$ and

$y = -3\sin\left(-\frac{\pi}{3}\right) = -3\left(-\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$ give us the Cartesian

coordinates $\left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$.

5. (a) $x = -4$ and $y = 4 \;\Rightarrow\; r = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$ and $\tan\theta = \frac{4}{-4} = -1 \quad [\theta = -\frac{\pi}{4} + n\pi]$. Since $(-4, 4)$ is in the second quadrant, the polar coordinates are (i) $\left(4\sqrt{2}, \frac{3\pi}{4}\right)$ and (ii) $\left(-4\sqrt{2}, \frac{7\pi}{4}\right)$.

(b) $x = 3$ and $y = 3\sqrt{3} \;\Rightarrow\; r = \sqrt{3^2 + \left(3\sqrt{3}\right)^2} = \sqrt{9 + 27} = 6$ and $\tan\theta = \frac{3\sqrt{3}}{3} = \sqrt{3} \quad [\theta = \frac{\pi}{3} + n\pi]$.

Since $\left(3, 3\sqrt{3}\right)$ is in the first quadrant, the polar coordinates are (i) $\left(6, \frac{\pi}{3}\right)$ and (ii) $\left(-6, \frac{4\pi}{3}\right)$.

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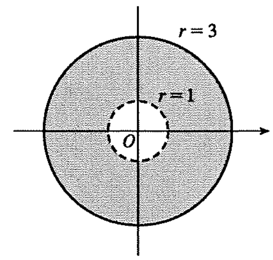
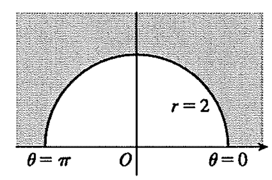
6. (a) $x = \sqrt{3}$ and $y = -1 \;\Rightarrow\; r = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$ and $\tan\theta = \dfrac{-1}{\sqrt{3}}$ $\left[\theta = -\frac{\pi}{6} + n\pi\right]$. Since $\left(\sqrt{3}, -1\right)$ is in the fourth quadrant, the polar coordinates are (i) $\left(2, \frac{11\pi}{6}\right)$ and (ii) $\left(-2, \frac{5\pi}{6}\right)$.

(b) $x = -6$ and $y = 0 \;\Rightarrow\; r = \sqrt{(-6)^2 + 0^2} = 6$ and $\tan\theta = \frac{0}{-6} = 0$ $[\theta = n\pi]$. Since $(-6, 0)$ is on the negative x-axis, the polar coordinates are (i) $(6, \pi)$ and (ii) $(-6, 0)$.

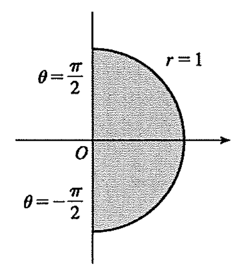
7. $1 < r \leq 3$. The curves $r = 1$ and $r = 3$ represent circles centered at O with radius 1 and 3, respectively. So $1 < r \leq 3$ represents the region outside the radius 1 circle and on or inside the radius 3 circle. Note that θ can take on any value.



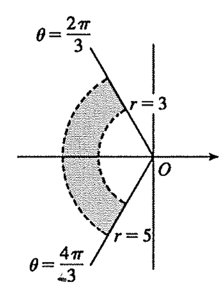
8. $r \geq 2,\ 0 \leq \theta \leq \pi$. This is the region on or outside the circle $r = 2$ in the first and second quadrants.



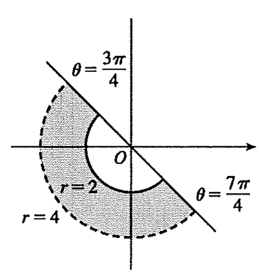
9. $0 \leq r \leq 1,\ -\pi/2 \leq \theta \leq \pi/2$. This is the region on or inside the circle $r = 1$ in the first and fourth quadrants.



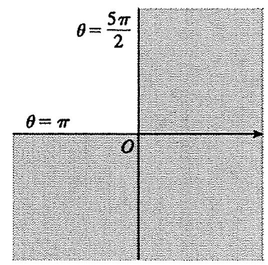
10. $3 < r < 5,\ 2\pi/3 \leq \theta \leq 4\pi/3$



11. $2 \leq r < 4, 3\pi/4 \leq \theta \leq 7\pi/4$



12. $r \geq 0,\ \pi \leq \theta \leq 5\pi/2$. This is the region in the third, fourth, and first quadrants including the origin and points on the negative x-axis and positive y-axis.



13. Converting the polar coordinates $\left(4, \frac{4\pi}{3}\right)$ and $\left(6, \frac{5\pi}{3}\right)$ to Cartesian coordinates gives us $\left(4\cos\frac{4\pi}{3}, 4\sin\frac{4\pi}{3}\right) = \left(-2, -2\sqrt{3}\right)$ and $\left(6\cos\frac{5\pi}{3}, 6\sin\frac{5\pi}{3}\right) = \left(3, -3\sqrt{3}\right)$. Now use the distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[3 - (-2)]^2 + \left[-3\sqrt{3} - \left(-2\sqrt{3}\right)\right]^2}$$
$$= \sqrt{5^2 + \left(-\sqrt{3}\right)^2} = \sqrt{25 + 3} = \sqrt{28} = 2\sqrt{7}$$

14. The points (r_1, θ_1) and (r_2, θ_2) in Cartesian coordinates are $(r_1\cos\theta_1, r_1\sin\theta_1)$ and $(r_2\cos\theta_2, r_2\sin\theta_2)$, respectively. The *square* of the distance between them is

$$(r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2$$
$$= \left(r_2^2\cos^2\theta_2 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\cos^2\theta_1\right) + \left(r_2^2\sin^2\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2 + r_1^2\sin^2\theta_1\right)$$
$$= r_1^2\left(\sin^2\theta_1 + \cos^2\theta_1\right) + r_2^2\left(\sin^2\theta_2 + \cos^2\theta_2\right) - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$$
$$= r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2,$$

so the distance between them is $\sqrt{r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2}$.

15. $r^2 = 5 \iff x^2 + y^2 = 5$, a circle of radius $\sqrt{5}$ centered at the origin.

16. $r = 4\sec\theta \iff \dfrac{r}{\sec\theta} = 4 \iff r\cos\theta = 4 \iff x = 4$, a vertical line.

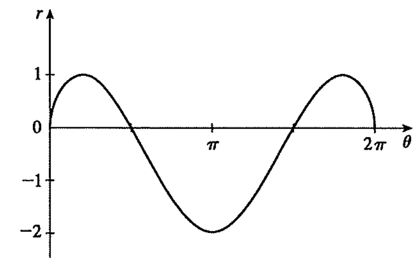
17. $r = 5\cos\theta \Rightarrow r^2 = 5r\cos\theta \iff x^2 + y^2 = 5x \iff x^2 - 5x + \frac{25}{4} + y^2 = \frac{25}{4} \iff \left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}$, a circle of radius $\frac{5}{2}$ centered at $\left(\frac{5}{2}, 0\right)$. The first two equations are actually equivalent since $r^2 = 5r\cos\theta \Rightarrow r(r - 5\cos\theta) = 0 \Rightarrow r = 0$ or $r = 5\cos\theta$. But $r = 5\cos\theta$ gives the point $r = 0$ (the pole) when $\theta = 0$. Thus, the equation $r = 5\cos\theta$ is equivalent to the compound condition ($r = 0$ or $r = 5\cos\theta$).

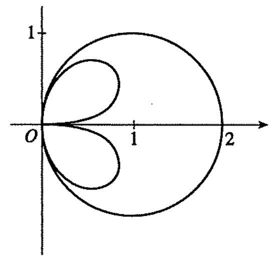
18. $\theta = \dfrac{\pi}{3} \Rightarrow \tan\theta = \tan\dfrac{\pi}{3} \Rightarrow \dfrac{y}{x} = \sqrt{3} \iff y = \sqrt{3}\,x$, a line through the origin.

19. $r^2\cos 2\theta = 1 \iff r^2(\cos^2\theta - \sin^2\theta) = 1 \iff (r\cos\theta)^2 - (r\sin\theta)^2 = 1 \iff x^2 - y^2 = 1$, a hyperbola centered at the origin with foci on the x-axis.

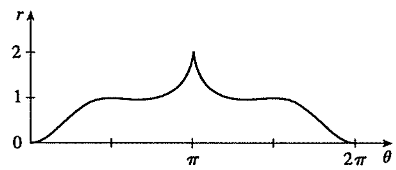
20. $r^2\sin 2\theta = 1 \iff r^2(2\sin\theta\,\cos\theta) = 1 \iff 2(r\cos\theta)(r\sin\theta) = 1 \iff 2xy = 1 \iff xy = \frac{1}{2}$, a hyperbola centered at the origin with foci on the line $y = x$.

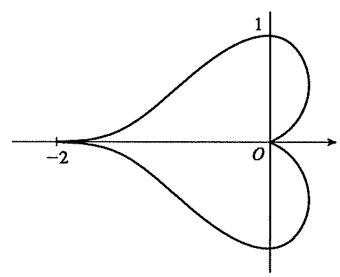
31. r has a maximum value of approximately 1 slightly before $\theta = \frac{\pi}{4}$ and slightly after $\theta = \frac{7\pi}{4}$. r has a minimum value of -2 when $\theta = \pi$. The graph touches the pole ($r = 0$) when $\theta = 0, \frac{\pi}{2}, \frac{3\pi}{2}$, and 2π. Since r is positive in the θ-intervals $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$, and negative in the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, the graph lies entirely in the first and fourth quadrants.



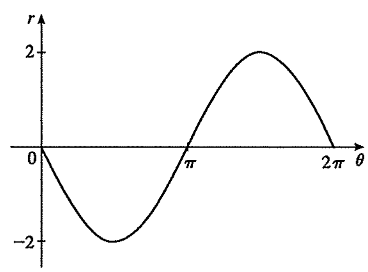


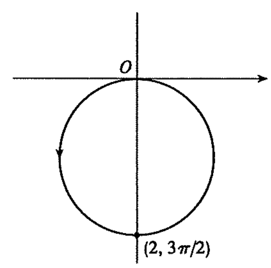
32. r increases from 0 to 1 (local max) in the interval $\left[0, \frac{\pi}{2}\right]$. It then decreases slightly, after which r increases to a maximum of 2 at $\theta = \pi$. The graph is symmetric about $\theta = \pi$, so the polar curve is symmetric about the polar axis.



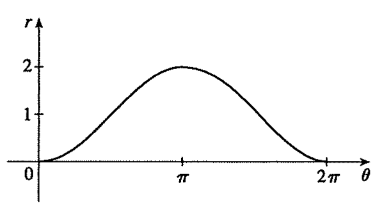


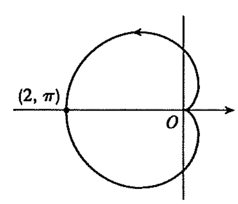
33. $r = -2\sin\theta$



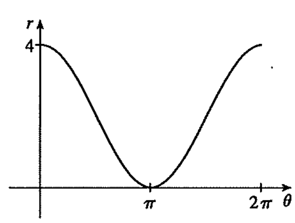


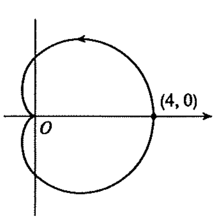
34. $r = 1 - \cos\theta$





35. $r = 2(1 + \cos\theta)$





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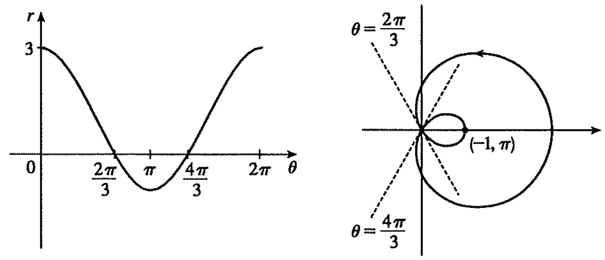
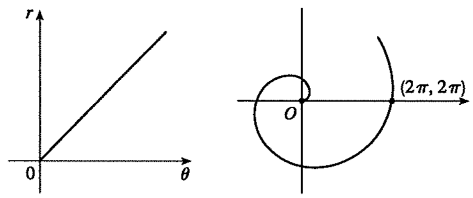
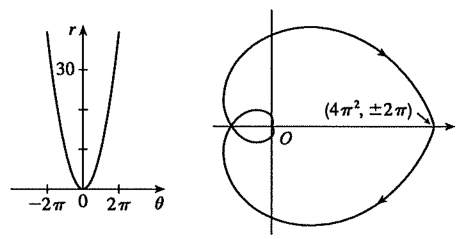
36. $r = 1 + 2\cos\theta$



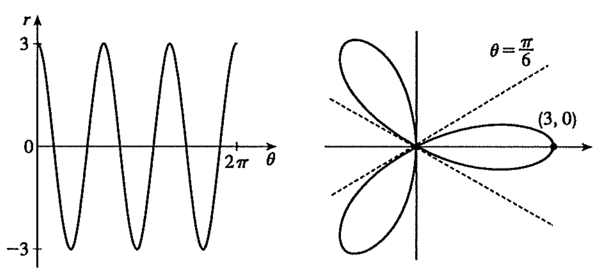
37. $r = \theta, \quad \theta \geq 0$



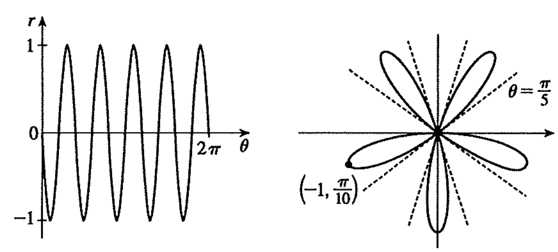
38. $r = \theta^2, \ -2\pi \leq \theta \leq 2\pi$



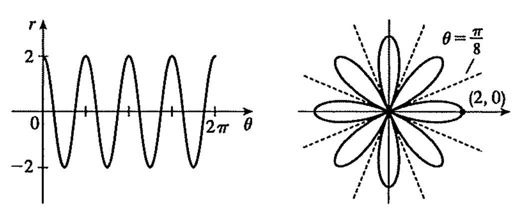
39. $r = 3\cos 3\theta$



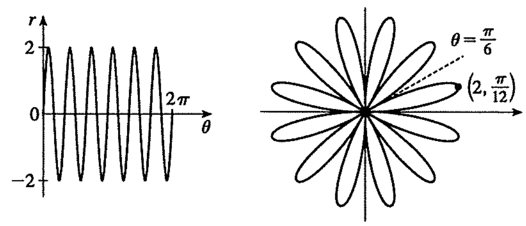
40. $r = -\sin 5\theta$



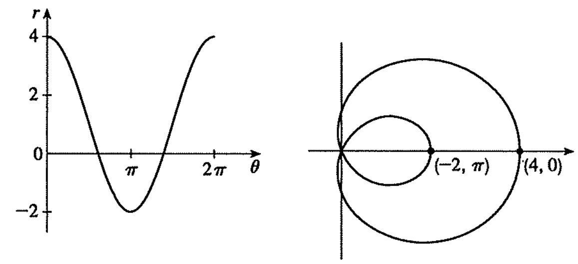
41. $r = 2\cos 4\theta$



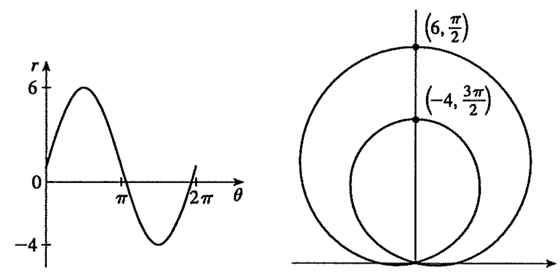
42. $r = 2\sin 6\theta$



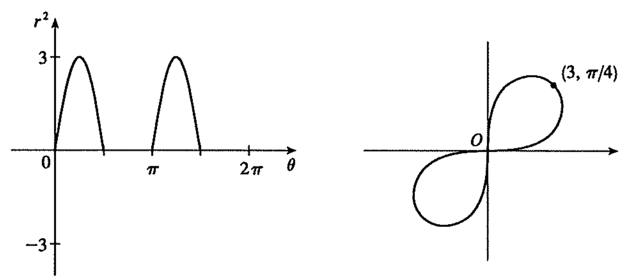
43. $r = 1 + 3\cos\theta$



44. $r = 1 + 5\sin\theta$



45. $r^2 = 9\sin 2\theta$



46. $r^2 = \cos 4\theta$