HW #9, Sec 9.1 Solutions

Sec 9.1

11. $y = x^3 \;\Rightarrow\; y' = 3x^2 \;\Rightarrow\; y'' = 6x.$

LHS $= x^2y'' - 6y = x^2 \cdot 6x - 6 \cdot x^3 = 6x^3 - 6x^3 = 0 =$ RHS, so $y = x^3$ **is** a solution of the differential equation.

12. $y = \ln x \;\Rightarrow\; y' = 1/x \;\Rightarrow\; y'' = -1/x^2.$

LHS $= x\,y'' - y' = x\left(-\frac{1}{x^2}\right) - \frac{1}{x} = -\frac{1}{x} - \frac{1}{x} = -\frac{2}{x} \neq 0$, so $y = \ln x$ **is not** a solution of the differential equation.

13. $y = -t\cos t - t \;\Rightarrow\; dy/dt = -t(-\sin t) + \cos t(-1) - 1 = t\sin t - \cos t - 1.$

LHS $= t\,\frac{dy}{dt} = t(t\sin t - \cos t - 1) = t^2\sin t - t\cos t - t = t^2\sin t + (-t\cos t - t) = t^2\sin t + y =$ RHS,

so y **is** a solution of the differential equation. Also, $y(\pi) = -\pi\cos\pi - \pi = -\pi(-1) - \pi = \pi - \pi = 0$, so the initial condition, $y(\pi) = 0$, is satisfied.

14. $y = 5e^{2x} + x \;\Rightarrow\; dy/dx = 10e^{2x} + 1.$

LHS $= \frac{dy}{dx} - 2y = 10e^{2x} + 1 - 2(5e^{2x} + x) = 1 - 2x =$ RHS, so y **is** a solution of the differential equation. Also, $y(0) = 5e^{2(0)} + 0 = 5$, so the initial condition, $y(0) = 5$, is satisfied.

15. (a) $y = e^{rx} \;\Rightarrow\; y' = re^{rx} \;\Rightarrow\; y'' = r^2e^{rx}$. Substituting these expressions into the differential equation $2y'' + y' - y = 0$, we get $2r^2e^{rx} + re^{rx} - e^{rx} = 0 \;\Rightarrow\; (2r^2 + r - 1)e^{rx} = 0 \;\Rightarrow$ $(2r - 1)(r + 1) = 0$ [since e^{rx} is never zero] $\Rightarrow\; r = \frac{1}{2}$ or -1.

(b) Let $r_1 = \frac{1}{2}$ and $r_2 = -1$, so we need to show that every member of the family of functions $y = ae^{x/2} + be^{-x}$ is a solution of the differential equation $2y'' + y' - y = 0$.

$y = ae^{x/2} + be^{-x} \;\Rightarrow\; y' = \frac{1}{2}ae^{x/2} - be^{-x} \;\Rightarrow\; y'' = \frac{1}{4}ae^{x/2} + be^{-x}.$

$$\begin{aligned}\text{LHS} = 2y'' + y' - y &= 2\left(\tfrac{1}{4}ae^{x/2} + be^{-x}\right) + \left(\tfrac{1}{2}ae^{x/2} - be^{-x}\right) - \left(ae^{x/2} + be^{-x}\right)\\ &= \tfrac{1}{2}ae^{x/2} + 2be^{-x} + \tfrac{1}{2}ae^{x/2} - be^{-x} - ae^{x/2} - be^{-x}\\ &= \left(\tfrac{1}{2}a + \tfrac{1}{2}a - a\right)e^{x/2} + (2b - b - b)e^{-x}\\ &= 0 = \text{RHS}\end{aligned}$$

16. (a) $y = \cos kt \;\Rightarrow\; y' = -k\sin kt \;\Rightarrow\; y'' = -k^2\cos kt$. Substituting these expressions into the differential equation $4y'' = -25y$, we get $4(-k^2\cos kt) = -25(\cos kt) \;\Rightarrow\; (25 - 4k^2)\cos kt = 0$ [for all t] $\Rightarrow\; 25 - 4k^2 = 0 \;\Rightarrow$ $k^2 = \frac{25}{4} \;\Rightarrow\; k = \pm\frac{5}{2}$.

(b) $y = A\sin kt + B\cos kt \;\Rightarrow\; y' = Ak\cos kt - Bk\sin kt \;\Rightarrow\; y'' = -Ak^2\sin kt - Bk^2\cos kt.$

The given differential equation $4y'' = -25y$ is equivalent to $4y'' + 25y = 0$. Thus,

$$\begin{aligned}\text{LHS} = 4y'' + 25y &= 4(-Ak^2\sin kt - Bk^2\cos kt) + 25(A\sin kt + B\cos kt)\\ &= -4Ak^2\sin kt - 4Bk^2\cos kt + 25A\sin kt + 25B\cos kt\\ &= (25 - 4k^2)A\sin kt + (25 - 4k^2)B\cos kt\\ &= 0 \quad \text{since } k^2 = \tfrac{25}{4}.\end{aligned}$$

17. (a) $y = \sin x \;\Rightarrow\; y' = \cos x \;\Rightarrow\; y'' = -\sin x$.

LHS $= y'' + y = -\sin x + \sin x = 0 \neq \sin x$, so $y = \sin x$ **is not** a solution of the differential equation.

(b) $y = \cos x \;\Rightarrow\; y' = -\sin x \;\Rightarrow\; y'' = -\cos x$.

LHS $= y'' + y = -\cos x + \cos x = 0 \neq \sin x$, so $y = \cos x$ **is not** a solution of the differential equation.

(c) $y = \frac{1}{2}x \sin x \;\Rightarrow\; y' = \frac{1}{2}(x\cos x + \sin x) \;\Rightarrow\; y'' = \frac{1}{2}(-x\sin x + \cos x + \cos x)$.

LHS $= y'' + y = \frac{1}{2}(-x\sin x + 2\cos x) + \frac{1}{2}x\sin x = \cos x \neq \sin x$, so $y = \frac{1}{2}x\sin x$ **is not** a solution of the differential equation.

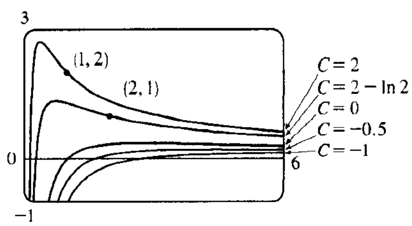
(d) $y = -\frac{1}{2}x\cos x \;\Rightarrow\; y' = -\frac{1}{2}(-x\sin x + \cos x) \;\Rightarrow\; y'' = -\frac{1}{2}(-x\cos x - \sin x - \sin x)$.

LHS $= y'' + y = -\frac{1}{2}(-x\cos x - 2\sin x) + \left(-\frac{1}{2}x\cos x\right) = \sin x =$ RHS, so $y = -\frac{1}{2}x\cos x$ **is** a solution of the differential equation.

18. (a) $y = \dfrac{\ln x + C}{x} \;\Rightarrow\; y' = \dfrac{x \cdot (1/x) - (\ln x + C)}{x^2} = \dfrac{1 - \ln x - C}{x^2}$.

$$\text{LHS} = x^2 y' + xy = x^2 \cdot \frac{1 - \ln x - C}{x^2} + x \cdot \frac{\ln x + C}{x}$$
$$= 1 - \ln x - C + \ln x + C = 1 = \text{RHS}, \text{ so } y \text{ is a solution of the differential equation.}$$

(b)



A few notes about the graph of $y = (\ln x + C)/x$:

(1) There is a vertical asymptote of $x = 0$.

(2) There is a horizontal asymptote of $y = 0$.

(3) $y = 0 \;\Rightarrow\; \ln x + C = 0 \;\Rightarrow\; x = e^{-C}$, so there is an x-intercept at e^{-C}.

(4) $y' = 0 \;\Rightarrow\; \ln x = 1 - C \;\Rightarrow\; x = e^{1-C}$, so there is a local maximum at $x = e^{1-C}$.

(c) $y(1) = 2 \;\Rightarrow\; 2 = \dfrac{\ln 1 + C}{1} \;\Rightarrow\; 2 = C$, so the solution is $y = \dfrac{\ln x + 2}{x}$ [shown in part (b)].

(d) $y(2) = 1 \;\Rightarrow\; 1 = \dfrac{\ln 2 + C}{2} \;\Rightarrow\; 2 + \ln 2 + C \;\Rightarrow\; C = 2 - \ln 2$, so the solution is $y = \dfrac{\ln x + 2 - \ln 2}{x}$ [shown in part (b)].

19. (a) Since the derivative $y' = -y^2$ is always negative (or 0, if $y = 0$), the function y must be decreasing (or equal to 0) on any interval on which it is defined.

(b) $y = \dfrac{1}{x + C} \;\Rightarrow\; y' = -\dfrac{1}{(x+C)^2}$. LHS $= y' = -\dfrac{1}{(x+C)^2} = -\left(\dfrac{1}{x+C}\right)^2 = -y^2 =$ RHS

(c) $y = 0$ is a solution of $y' = -y^2$ that is not a member of the family in part (b).

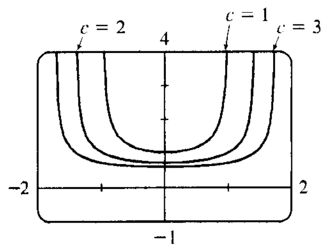
(d) If $y(x) = \dfrac{1}{x+C}$, then $y(0) = \dfrac{1}{0+C} = \dfrac{1}{C}$. Since $y(0) = 0.5$, $\dfrac{1}{C} = \dfrac{1}{2} \;\Rightarrow\; C = 2$, so $y = \dfrac{1}{x+2}$.

Sec 9.1

20. (a) If x is close to 0, then xy^3 is close to 0, and hence, y' is close to 0. Thus, the graph of y must have a tangent line that is nearly horizontal. If x is large, then xy^3 is large, and the graph of y must have a tangent line that is nearly vertical. (In both cases, we assume reasonable values for y.)

(b) $y = (c - x^2)^{-1/2} \Rightarrow y' = x(c - x^2)^{-3/2}$. RHS $= xy^3 = x[(c - x^2)^{-1/2}]^3 = x(c - x^2)^{-3/2} = y' =$ LHS

(c)



When x is close to 0, y' is also close to 0. As x gets larger, so does $|y'|$.

(d) $y(0) = (c - 0)^{-1/2} = 1/\sqrt{c}$ and $y(0) = 2 \Rightarrow \sqrt{c} = \frac{1}{2} \Rightarrow c = \frac{1}{4}$, so $y = \left(\frac{1}{4} - x^2\right)^{-1/2}$.

21. (a) $\dfrac{dP}{dt} = 1.2P\left(1 - \dfrac{P}{4200}\right)$. Now $\dfrac{dP}{dt} > 0 \Rightarrow 1 - \dfrac{P}{4200} > 0$ [assuming that $P > 0$] $\Rightarrow \dfrac{P}{4200} < 1 \Rightarrow P < 4200 \Rightarrow$ the population is increasing for $0 < P < 4200$.

(b) $\dfrac{dP}{dt} < 0 \Rightarrow P > 4200$

(c) $\dfrac{dP}{dt} = 0 \Rightarrow P = 4200$ or $P = 0$

22. (a) $\dfrac{dv}{dt} = -v[v^2 - (1 + a)v + a] = -v(v - a)(v - 1)$, so $\dfrac{dv}{dt} = 0 \Leftrightarrow v = 0, a,$ or 1.

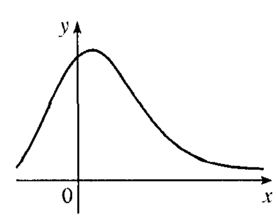
(b) With $0 < a < 1$, $dv/dt = -v(v - a)(v - 1) > 0 \Leftrightarrow v < 0$ or $a < v < 1$, so v is increasing on $(-\infty, 0)$ and $(a, 1)$.

(c) With $0 < a < 1$, $dv/dt = -v(v - a)(v - 1) < 0 \Leftrightarrow 0 < v < a$ or $v > 1$, so v is decreasing on $(0, a)$ and $(1, \infty)$.

23. (a) This function is increasing *and* also decreasing. But $dy/dt = e^t(y - 1)^2 \geq 0$ for all t, implying that the graph of the solution of the differential equation cannot be decreasing on any interval.

(b) When $y = 1$, $dy/dt = 0$, but the graph does not have a horizontal tangent line.

24. The graph for this exercise is shown in the figure at the right.



A. $y' = 1 + xy > 1$ for points in the first quadrant, but we can see that $y' < 0$ for some points in the first quadrant.

B. $y' = -2xy = 0$ when $x = 0$, but we can see that $y' > 0$ for $x = 0$. Thus, equations A and B are incorrect, so the correct equation is C.

C. $y' = 1 - 2xy$ seems reasonable since:

(1) When $x = 0$, y' could be 1.

(2) When $x < 0$, y' could be greater than 1.

(3) Solving $y' = 1 - 2xy$ for y gives us $y = \dfrac{1 - y'}{2x}$. If y' takes on small negative values, then as $x \to \infty$, $y \to 0^+$, as shown in the figure.

25. (a) $y' = 1 + x^2 + y^2 \geq 1$ and $y' \to \infty$ as $x \to \infty$. The only curve satisfying these conditions is labeled III.

(b) $y' = xe^{-x^2-y^2} > 0$ if $x > 0$ and $y' < 0$ if $x < 0$. The only curve with negative tangent slopes when $x < 0$ and positive tangent slopes when $x > 0$ is labeled I.

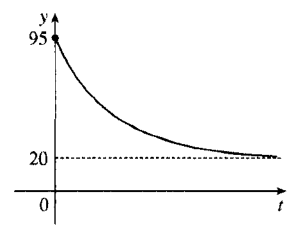
(c) $y' = \dfrac{1}{1+e^{x^2+y^2}} > 0$ and $y' \to 0$ as $x \to \infty$. The only curve satisfying these conditions is labeled IV.

(d) $y' = \sin(xy)\cos(xy) = 0$ if $y = 0$, which is the solution graph labeled II.

26. (a) The coffee cools most quickly as soon as it is removed from the heat source. The rate of cooling decreases toward 0 since the coffee approaches room temperature.

(b) $\dfrac{dy}{dt} = k(y - R)$, where k is a proportionality constant, y is the temperature of the coffee, and R is the room temperature. The initial condition is $y(0) = 95°\text{C}$. The answer and the model support each other because as y approaches R, dy/dt approaches 0, so the model seems appropriate.

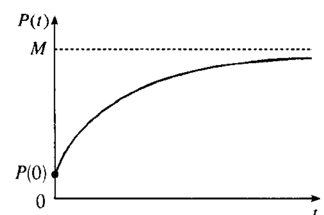
(c)



27. (a) P increases most rapidly at the beginning, since there are usually many simple, easily-learned sub-skills associated with learning a skill. As t increases, we would expect dP/dt to remain positive, but decrease. This is because as time progresses, the only points left to learn are the more difficult ones.

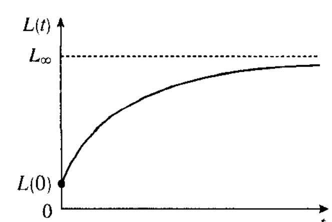
(b) $\dfrac{dP}{dt} = k(M - P)$ is always positive, so the level of performance P is increasing. As P gets close to M, dP/dt gets close to 0; that is, the performance levels off, as explained in part (a).

(c)



28. (a) $\dfrac{dL}{dt} = k(L_\infty - L)$. Assuming $L_\infty > L$, we have $k > 0$ and $dL/dt > 0$ for all t.

(b)



29. If $c(t) = c_s\left(1 - e^{-\alpha t^{1-b}}\right) = c_s - c_s e^{-\alpha t^{1-b}}$ for $t > 0$, where $k > 0$, $c_s > 0$, $0 < b < 1$, and $\alpha = k/(1-b)$, then

$$\frac{dc}{dt} = c_s\left[0 - e^{-\alpha t^{1-b}} \cdot \frac{d}{dt}\left(-\alpha t^{1-b}\right)\right] = -c_s e^{-\alpha t^{1-b}} \cdot (-\alpha)(1-b)t^{-b} = \frac{\alpha(1-b)}{t^b}\, c_s\, e^{-\alpha t^{1-b}} = \frac{k}{t^b}(c_s - c).$$

The equation for c indicates that as t increases, c approaches c_s. The differential equation indicates that as t increases, the rate of increase of c decreases steadily and approaches 0 as c approaches c_s.