

17 □ SECOND-ORDER DIFFERENTIAL EQUATIONS

17.1 Second-Order Linear Equations

1. The auxiliary equation is $r^2 - r - 6 = 0 \Rightarrow (r - 3)(r + 2) = 0 \Rightarrow r = 3, r = -2$. Then by (8) the general solution is $y = c_1 e^{3x} + c_2 e^{-2x}$.
2. The auxiliary equation is $r^2 - 6r + 9 = 0 \Rightarrow (r - 3)^2 = 0 \Rightarrow r = 3$. Then by (10), the general solution is $y = c_1 e^{3x} + c_2 x e^{3x}$.
3. The auxiliary equation is $r^2 + 2 = 0 \Rightarrow r = \pm \sqrt{2}i$. Then by (11) the general solution is $y = e^{0x} (c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)) = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)$.
4. The auxiliary equation is $r^2 + r - 12 = 0 \Rightarrow (r - 3)(r + 4) = 0 \Rightarrow r = 3, r = -4$. Then by (8) the general solution is $y = c_1 e^{3x} + c_2 e^{-4x}$.
5. The auxiliary equation is $4r^2 + 4r + 1 = 0 \Rightarrow (2r + 1)^2 = 0 \Rightarrow r = -\frac{1}{2}$. Then by (10), the general solution is $y = c_1 e^{-x/2} + c_2 x e^{-x/2}$.
6. The auxiliary equation is $9r^2 + 4 = 0 \Rightarrow r^2 = -\frac{4}{9} \Rightarrow r = \pm \frac{2}{3}i$, so the general solution is $y = e^{0x} [c_1 \cos(\frac{2}{3}x) + c_2 \sin(\frac{2}{3}x)] = c_1 \cos(\frac{2}{3}x) + c_2 \sin(\frac{2}{3}x)$.
7. The auxiliary equation is $3r^2 - 4r = r(3r - 4) = 0 \Rightarrow r = 0, r = \frac{4}{3}$, so $y = c_1 e^{0x} + c_2 e^{4x/3} = c_1 + c_2 e^{4x/3}$.
8. The auxiliary equation is $r^2 - 1 = (r - 1)(r + 1) = 0 \Rightarrow r = 1, r = -1$. Then the general solution is $y = c_1 e^x + c_2 e^{-x}$.
9. The auxiliary equation is $r^2 - 4r + 13 = 0 \Rightarrow r = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$, so $y = e^{2x} (c_1 \cos 3x + c_2 \sin 3x)$.
10. The auxiliary equation is $3r^2 + 4r - 3 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{52}}{6} = \frac{-2 \pm \sqrt{13}}{3}$, so $y = c_1 e^{(-2+\sqrt{13})x/3} + c_2 e^{(-2-\sqrt{13})x/3}$.
11. The auxiliary equation is $2r^2 + 2r - 1 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{12}}{4} = \frac{-1 \pm \sqrt{3}}{2}$, so $y = c_1 e^{(-1+\sqrt{3})t/2} + c_2 e^{(-1-\sqrt{3})t/2}$.
12. The auxiliary equation is $r^2 + 6r + 34 = 0 \Rightarrow r = \frac{-6 \pm \sqrt{-100}}{2} = -3 \pm 5i$, so $R = e^{-3t} (c_1 \cos 5t + c_2 \sin 5t)$.

13. The auxiliary equation is $3r^2 + 4r + 3 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{-20}}{6} = -\frac{2}{3} \pm \frac{\sqrt{5}}{3}i$, so

$$V = e^{-2t/3} \left[c_1 \cos\left(\frac{\sqrt{5}}{3}t\right) + c_2 \sin\left(\frac{\sqrt{5}}{3}t\right) \right].$$

14. The auxiliary equation is $4r^2 - 4r + 1 = (2r - 1)^2 = 0 \Rightarrow r = \frac{1}{2}$, so

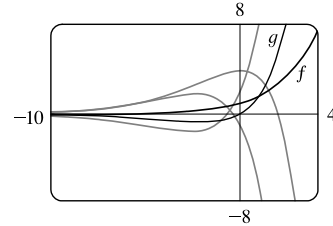
the general solution is $y = c_1 e^{x/2} + c_2 x e^{x/2}$. We graph the basic solutions

$$f(x) = e^{x/2}, g(x) = x e^{x/2} \text{ as well as } y = 2e^{x/2} + 3x e^{x/2},$$

$$y = -e^{x/2} - 3x e^{x/2}, \text{ and } y = 4e^{x/2} - 2x e^{x/2}. \text{ The graphs are all}$$

asymptotic to the x -axis as $x \rightarrow -\infty$, and as $x \rightarrow \infty$ the solutions

approach $\pm\infty$.



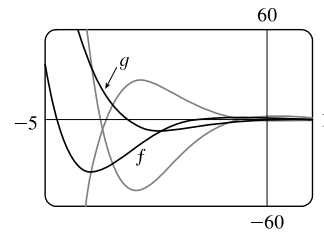
15. The auxiliary equation is $r^2 + 2r + 2 = 0 \Rightarrow$

$$r = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i, \text{ so the general solution is}$$

$y = e^{-x} (c_1 \cos x + c_2 \sin x)$. We graph the basic solutions

$$f(x) = e^{-x} \cos x, g(x) = e^{-x} \sin x \text{ as well as}$$

$y = e^{-x} (-\cos x - 2 \sin x)$ and $y = e^{-x} (2 \cos x + 3 \sin x)$. All the solutions oscillate with amplitudes that become arbitrarily large as $x \rightarrow -\infty$ and the solutions are asymptotic to the x -axis as $x \rightarrow \infty$.



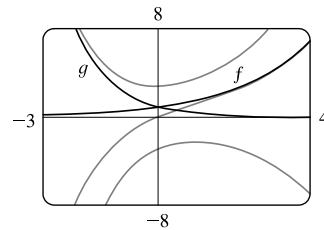
16. The auxiliary equation is $2r^2 + r - 1 = (2r - 1)(r + 1) = 0 \Rightarrow$

$$r = \frac{1}{2}, r = -1, \text{ so the general solution is } y = c_1 e^{x/2} + c_2 e^{-x}. \text{ We graph}$$

the basic solutions $f(x) = e^{x/2}, g(x) = e^{-x}$ as well as $y = 2e^{x/2} + e^{-x}$,

$$y = -e^{x/2} - 2e^{-x}, \text{ and } y = e^{x/2} - e^{-x}. \text{ Each solution consists of a single}$$

continuous curve that approaches either 0 or $\pm\infty$ as $x \rightarrow \pm\infty$.



17. $r^2 + 3 = 0 \Rightarrow r = \pm\sqrt{3}i$ and the general solution is

$$y = e^{0x} [c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)] = c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x). \text{ Then } y(0) = 1 \Rightarrow c_1 = 1 \text{ and, since}$$

$$y' = -\sqrt{3}c_1 \sin(\sqrt{3}x) + \sqrt{3}c_2 \cos(\sqrt{3}x), y'(0) = 3 \Rightarrow \sqrt{3}c_2 = 3 \Rightarrow c_2 = \frac{3}{\sqrt{3}} = \sqrt{3}, \text{ so the solution to the}$$

initial-value problem is $y = \cos(\sqrt{3}x) + \sqrt{3} \sin(\sqrt{3}x)$.

18. $r^2 - 2r - 3 = (r - 3)(r + 1) = 0$, so $r = 3, r = -1$ and the general solution is $y = c_1 e^{3x} + c_2 e^{-x}$. Then

$$y' = 3c_1 e^{3x} - c_2 e^{-x}, \text{ so } y(0) = 2 \Rightarrow c_1 + c_2 = 2 \text{ and } y'(0) = 2 \Rightarrow 3c_1 - c_2 = 2, \text{ giving } c_1 = 1 \text{ and } c_2 = 1. \text{ Thus}$$

the solution to the initial-value problem is $y = e^{3x} + e^{-x}$.

19. $9r^2 + 12r + 4 = (3r + 2)^2 = 0 \Rightarrow r = -\frac{2}{3}$ and the general solution is $y = c_1 e^{-2x/3} + c_2 x e^{-2x/3}$. Then $y(0) = 1 \Rightarrow$

$$c_1 = 1 \text{ and, since } y' = -\frac{2}{3}c_1 e^{-2x/3} + c_2 (1 - \frac{2}{3}x) e^{-2x/3}, y'(0) = 0 \Rightarrow -\frac{2}{3}c_1 + c_2 = 0, \text{ so } c_2 = \frac{2}{3} \text{ and the solution to}$$

the initial-value problem is $y = e^{-2x/3} + \frac{2}{3}x e^{-2x/3}$.

20. $3r^2 - 2r - 1 = (3r + 1)(r - 1) = 0 \Rightarrow r = -\frac{1}{3}, r = 1$ and the general solution is $y = c_1 e^{-x/3} + c_2 e^x$. Then $y' = -\frac{1}{3}c_1 e^{-x/3} + c_2 e^x$, so $y(0) = 0 \Rightarrow c_1 + c_2 = 0$ and $y'(0) = -4 \Rightarrow -\frac{1}{3}c_1 + c_2 = -4$, giving $c_1 = 3$ and $c_2 = -3$. Thus the solution to the initial-value problem is $y = 3e^{-x/3} - 3e^x$.
21. $r^2 - 6r + 10 = 0 \Rightarrow r = 3 \pm i$ and the general solution is $y = e^{3x}(c_1 \cos x + c_2 \sin x)$. Then $2 = y(0) = c_1$ and $3 = y'(0) = c_2 + 3c_1 \Rightarrow c_2 = -3$ and the solution to the initial-value problem is $y = e^{3x}(2 \cos x - 3 \sin x)$.
22. $4r^2 - 20r + 25 = (2r - 5)^2 = 0 \Rightarrow r = \frac{5}{2}$ and the general solution is $y = c_1 e^{5x/2} + c_2 x e^{5x/2}$. Then $2 = y(0) = c_1$ and $-3 = y'(0) = \frac{5}{2}c_1 + c_2 \Rightarrow c_2 = -8$. The solution to the initial-value problem is $y = 2e^{5x/2} - 8xe^{5x/2}$.
23. $r^2 - r - 12 = (r - 4)(r + 3) = 0 \Rightarrow r = 4, r = -3$ and the general solution is $y = c_1 e^{4x} + c_2 e^{-3x}$. Then $0 = y(1) = c_1 e^4 + c_2 e^{-3}$ and $1 = y'(1) = 4c_1 e^4 - 3c_2 e^{-3}$ so $c_1 = \frac{1}{7}e^{-4}$, $c_2 = -\frac{1}{7}e^3$ and the solution to the initial-value problem is $y = \frac{1}{7}e^{-4}e^{4x} - \frac{1}{7}e^3e^{-3x} = \frac{1}{7}e^{4x-4} - \frac{1}{7}e^{3-3x}$.
24. $4r^2 + 4r + 3 = 0 \Rightarrow r = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}i$ and the general solution is $y = e^{-x/2} \left(c_1 \cos \frac{\sqrt{2}}{2}x + c_2 \sin \frac{\sqrt{2}}{2}x \right)$. Then $0 = y(0) = c_1$ and $1 = y'(0) = \frac{\sqrt{2}}{2}c_2 - \frac{1}{2}c_1 \Rightarrow c_2 = \sqrt{2}$ and the solution to the initial-value problem is $y = e^{-x/2} \left(0 + \sqrt{2} \sin \frac{\sqrt{2}}{2}x \right) = \sqrt{2}e^{-x/2} \sin \frac{\sqrt{2}}{2}x$.
25. $r^2 + 16 = 0 \Rightarrow r = \pm 4i$ and the general solution is $y = c_1 \cos 4x + c_2 \sin 4x$. Then $-3 = y(0) = c_1$ and $2 = y(\pi/8) = c_2$, so the solution of the boundary-value problem is $y = -3 \cos 4x + 2 \sin 4x$.
26. $r^2 + 6r = r(r + 6) = 0 \Rightarrow r = 0, r = -6$ and the general solution is $y = c_1 + c_2 e^{-6x}$. Then $1 = y(0) = c_1 + c_2$ and $0 = y(1) = c_1 + c_2 e^{-6}$ so $c_1 = \frac{1}{1 - e^6}$, $c_2 = -\frac{e^6}{1 - e^6}$. The solution of the boundary-value problem is $y = \frac{1}{1 - e^6} - \frac{e^6}{1 - e^6} \cdot e^{-6x} = \frac{1}{1 - e^6} - \frac{e^{6-6x}}{1 - e^6}$.
27. $r^2 + 4r + 4 = (r + 2)^2 = 0 \Rightarrow r = -2$ and the general solution is $y = c_1 e^{-2x} + c_2 x e^{-2x}$. Then $2 = y(0) = c_1$ and $0 = y(1) = c_1 e^{-2} + c_2 e^{-2}$ so $c_2 = -2$, and the solution of the boundary-value problem is $y = 2e^{-2x} - 2xe^{-2x}$.
28. $r^2 - 8r + 17 = 0 \Rightarrow r = 4 \pm i$ and the general solution is $y = e^{4x}(c_1 \cos x + c_2 \sin x)$. But $3 = y(0) = c_1$ and $2 = y(\pi) = -c_1 e^{4\pi} \Rightarrow c_1 = -2/e^{4\pi}$, so there is no solution.
29. $r^2 - r = r(r - 1) = 0 \Rightarrow r = 0, r = 1$ and the general solution is $y = c_1 + c_2 e^x$. Then $1 = y(0) = c_1 + c_2$ and $2 = y(1) = c_1 + c_2 e$ so $c_1 = \frac{e - 2}{e - 1}$, $c_2 = \frac{1}{e - 1}$. The solution of the boundary-value problem is $y = \frac{e - 2}{e - 1} + \frac{e^x}{e - 1}$.
30. $4r^2 - 4r + 1 = (2r - 1)^2 = 0 \Rightarrow r = \frac{1}{2}$ and the general solution is $y = c_1 e^{x/2} + c_2 x e^{x/2}$. Then $4 = y(0) = c_1$ and $0 = y(2) = c_1 e + 2c_2 e \Rightarrow c_2 = -2$. The solution of the boundary-value problem is $y = 4e^{x/2} - 2xe^{x/2}$.

31. $r^2 + 4r + 20 = 0 \Rightarrow r = -2 \pm 4i$ and the general solution is $y = e^{-2x}(c_1 \cos 4x + c_2 \sin 4x)$. But $1 = y(0) = c_1$ and $2 = y(\pi) = c_1 e^{-2\pi} \Rightarrow c_1 = 2e^{2\pi}$, so there is no solution.

32. $r^2 + 4r + 20 = 0 \Rightarrow r = -2 \pm 4i$ and the general solution is $y = e^{-2x}(c_1 \cos 4x + c_2 \sin 4x)$. But $1 = y(0) = c_1$ and $e^{-2\pi} = y(\pi) = c_1 e^{-2\pi} \Rightarrow c_1 = 1$, so c_2 can vary and the solution of the boundary-value problem is $y = e^{-2x}(\cos 4x + c \sin 4x)$, where c is any constant.

33. (a) Case 1 ($\lambda = 0$): $y'' + \lambda y = 0 \Rightarrow y'' = 0$ which has an auxiliary equation $r^2 = 0 \Rightarrow r = 0 \Rightarrow y = c_1 + c_2 x$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 L \Rightarrow c_1 = c_2 = 0$. Thus $y = 0$.

Case 2 ($\lambda < 0$): $y'' + \lambda y = 0$ has auxiliary equation $r^2 = -\lambda \Rightarrow r = \pm\sqrt{-\lambda}$ [distinct and real since $\lambda < 0$] $\Rightarrow y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ where $y(0) = 0$ and $y(L) = 0$. Thus $0 = y(0) = c_1 + c_2$ (*) and

$$0 = y(L) = c_1 e^{\sqrt{-\lambda}L} + c_2 e^{-\sqrt{-\lambda}L} \quad (\dagger).$$

Multiplying (*) by $e^{\sqrt{-\lambda}L}$ and subtracting (†) gives $c_2(e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}) = 0 \Rightarrow c_2 = 0$ and thus $c_1 = 0$ from (*).

Thus $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.

(b) $y'' + \lambda y = 0$ has an auxiliary equation $r^2 + \lambda = 0 \Rightarrow r = \pm i\sqrt{\lambda} \Rightarrow y = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 \sin \sqrt{\lambda}L$ since $c_1 = 0$. Since we cannot have a trivial solution, $c_2 \neq 0$ and thus $\sin \sqrt{\lambda}L = 0 \Rightarrow \sqrt{\lambda}L = n\pi$ where n is an integer $\Rightarrow \lambda = n^2\pi^2/L^2$ and $y = c_2 \sin(n\pi x/L)$ where n is an integer.

34. The auxiliary equation is $ar^2 + br + c = 0$. If $b^2 - 4ac > 0$, then any solution is of the form $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ where

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \text{ But } a, b, \text{ and } c \text{ are all positive so both } r_1 \text{ and } r_2 \text{ are negative and}$$

$\lim_{x \rightarrow \infty} y(x) = 0$. If $b^2 - 4ac = 0$, then any solution is of the form $y(x) = c_1 e^{rx} + c_2 x e^{rx}$ where $r = -b/(2a) < 0$

since a, b are positive. Hence $\lim_{x \rightarrow \infty} y(x) = 0$. Finally if $b^2 - 4ac < 0$, then any solution is of the form

$$y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \text{ where } \alpha = -b/(2a) < 0 \text{ since } a \text{ and } b \text{ are positive. Thus } \lim_{x \rightarrow \infty} y(x) = 0.$$

35. (a) $r^2 - 2r + 2 = 0 \Rightarrow r = 1 \pm i$ and the general solution is $y = e^x(c_1 \cos x + c_2 \sin x)$. If $y(a) = c$ and $y(b) = d$ then

$$e^a(c_1 \cos a + c_2 \sin a) = c \Rightarrow c_1 \cos a + c_2 \sin a = ce^{-a} \text{ and } e^b(c_1 \cos b + c_2 \sin b) = d \Rightarrow$$

$c_1 \cos b + c_2 \sin b = de^{-b}$. This gives a linear system in c_1 and c_2 which has a unique solution if the lines are not parallel.

If the lines are not vertical or horizontal, we have parallel lines if $\cos a = k \cos b$ and $\sin a = k \sin b$ for some nonzero

$$\text{constant } k \text{ or } \frac{\cos a}{\cos b} = k = \frac{\sin a}{\sin b} \Rightarrow \frac{\sin a}{\cos a} = \frac{\sin b}{\cos b} \Rightarrow \tan a = \tan b \Rightarrow b - a = n\pi, n \text{ any integer. (Note that}$$

none of $\cos a, \cos b, \sin a, \sin b$ are zero.) If the lines are both horizontal then $\cos a = \cos b = 0 \Rightarrow b - a = n\pi$, and

similarly vertical lines means $\sin a = \sin b = 0 \Rightarrow b - a = n\pi$. Thus the system has a unique solution if $b - a \neq n\pi$.

(b) The linear system has no solution if the lines are parallel but not identical. From part (a) the lines are parallel if

$$b - a = n\pi. \text{ If the lines are not horizontal, they are identical if } ce^{-a} = kde^{-b} \Rightarrow \frac{ce^{-a}}{de^{-b}} = k = \frac{\cos a}{\cos b} \Rightarrow$$

$$\frac{c}{d} = e^{a-b} \frac{\cos a}{\cos b}. \text{ (If } d = 0 \text{ then } c = 0 \text{ also.) If they are horizontal then } \cos b = 0, \text{ but } k = \frac{\sin a}{\sin b} \text{ also (and } \sin b \neq 0 \text{) so}$$

$$\text{we require } \frac{c}{d} = e^{a-b} \frac{\sin a}{\sin b}. \text{ Thus the system has no solution if } b - a = n\pi \text{ and } \frac{c}{d} \neq e^{a-b} \frac{\cos a}{\cos b} \text{ unless } \cos b = 0, \text{ in}$$

$$\text{which case } \frac{c}{d} \neq e^{a-b} \frac{\sin a}{\sin b}.$$

(c) The linear system has infinitely many solution if the lines are identical (and necessarily parallel). From part (b) this occurs

$$\text{when } b - a = n\pi \text{ and } \frac{c}{d} = e^{a-b} \frac{\cos a}{\cos b} \text{ unless } \cos b = 0, \text{ in which case } \frac{c}{d} = e^{a-b} \frac{\sin a}{\sin b}.$$

17.2 Nonhomogeneous Linear Equations

1. The auxiliary equation is $r^2 + 2r - 8 = (r - 2)(r + 4) = 0 \Rightarrow r = 2, r = -4$, so the complementary solution is

$$y_c(x) = c_1 e^{2x} + c_2 e^{-4x}. \text{ We try the particular solution } y_p(x) = Ax^2 + Bx + C, \text{ so } y'_p = 2Ax + B \text{ and } y''_p = 2A.$$

Substituting into the differential equation, we have $(2A) + 2(2Ax + B) - 8(Ax^2 + Bx + C) = 1 - 2x^2$ or

$$-8Ax^2 + (4A - 8B)x + (2A + 2B - 8C) = -2x^2 + 1. \text{ Comparing coefficients gives } -8A = -2 \Rightarrow$$

$$A = \frac{1}{4}, 4A - 8B = 0 \Rightarrow B = \frac{1}{8}, \text{ and } 2A + 2B - 8C = 1 \Rightarrow C = -\frac{1}{32}, \text{ so the general solution is}$$

$$y(x) = y_c(x) + y_p(x) = c_1 e^{2x} + c_2 e^{-4x} + \frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{32}.$$

2. The auxiliary equation is $r^2 - 3r = r(r - 3) = 0 \Rightarrow r = 0, r = 3$, so the complementary solution

$$\text{is } y_c(x) = c_1 + c_2 e^{3x}. \text{ We try the particular solution } y_p(x) = A \cos 2x + B \sin 2x, \text{ so}$$

$$y'_p = -2A \sin 2x + 2B \cos 2x \text{ and } y''_p = -4A \cos 2x - 4B \sin 2x. \text{ Substitution into the differential}$$

$$\text{equation gives } (-4A \cos 2x - 4B \sin 2x) - 3(-2A \sin 2x + 2B \cos 2x) = \sin 2x \Rightarrow$$

$$(-4A - 6B) \cos 2x + (6A - 4B) \sin 2x = \sin 2x. \text{ Then } -4A - 6B = 0 \text{ and } 6A - 4B = 1 \Rightarrow A = \frac{3}{26} \text{ and } B = -\frac{1}{13}.$$

$$\text{Thus the general solution is } y(x) = y_c(x) + y_p(x) = c_1 + c_2 e^{3x} + \frac{3}{26} \cos 2x - \frac{1}{13} \sin 2x.$$

3. The auxiliary equation is $9r^2 + 1 = 0$ with roots $r = \pm \frac{1}{3}i$, so the complementary solution is

$$y_c(x) = c_1 \cos(x/3) + c_2 \sin(x/3). \text{ Try the particular solution } y_p(x) = Ae^{2x}, \text{ so } y'_p = 2Ae^{2x} \text{ and } y''_p = 4Ae^{2x}.$$

$$\text{Substitution into the differential equation gives } 9(4Ae^{2x}) + (Ae^{2x}) = e^{2x} \text{ or } 37Ae^{2x} = e^{2x}. \text{ Thus } 37A = 1 \Rightarrow A = \frac{1}{37}$$

$$\text{and the general solution is } y(x) = y_c(x) + y_p(x) = c_1 \cos(x/3) + c_2 \sin(x/3) + \frac{1}{37}e^{2x}.$$

4. The auxiliary equation is $r^2 - 2r + 2 = 0$ with roots $r = 1 \pm i$, so the complementary solution is

$$y_c(x) = e^x(c_1 \cos x + c_2 \sin x). \text{ Try the particular solution } y_p(x) = Ax + B + Ce^x, \text{ so } y'_p = A + Ce^x \text{ and } y''_p = Ce^x.$$

$$\text{Substitution into the differential equation gives } (Ce^x) - 2(A + Ce^x) + 2(Ax + B + Ce^x) = x + e^x \Rightarrow$$

$2Ax + (-2A + 2B) + Ce^x = x + e^x$. Comparing coefficients, we have $2A = 1 \Rightarrow A = \frac{1}{2}$, $-2A + 2B = 0 \Rightarrow B = \frac{1}{2}$, and $C = 1$, so the general solution is $y(x) = y_c(x) + y_p(x) = e^x(c_1 \cos x + c_2 \sin x) + \frac{1}{2}x + \frac{1}{2} + e^x$.

5. The auxiliary equation is $r^2 - 4r + 5 = 0$ with roots $r = 2 \pm i$, so the complementary solution is

$y_c(x) = e^{2x}(c_1 \cos x + c_2 \sin x)$. Try $y_p(x) = Ae^{-x}$, so $y'_p = -Ae^{-x}$ and $y''_p = Ae^{-x}$. Substitution gives $Ae^{-x} - 4(-Ae^{-x}) + 5(Ae^{-x}) = e^{-x} \Rightarrow 10Ae^{-x} = e^{-x} \Rightarrow A = \frac{1}{10}$. Thus the general solution is $y(x) = e^{2x}(c_1 \cos x + c_2 \sin x) + \frac{1}{10}e^{-x}$.

6. The auxiliary equation is $r^2 - 4r + 4 = (r - 2)^2 = 0 \Rightarrow r = 2$, so the complementary solution is

$y_c(x) = c_1e^{2x} + c_2xe^{2x}$. For $y'' - 4y' + 4y = x$ try $y_{p1}(x) = Ax + B$. Then $y'_{p1} = A$ and $y''_{p1} = 0$, and substitution into the differential equation gives $0 - 4A + 4(Ax + B) = x$ or $4Ax + (4B - 4A) = x$, so $4A = 1 \Rightarrow A = \frac{1}{4}$ and

$4B - 4A = 0 \Rightarrow B = \frac{1}{4}$. Thus $y_{p1}(x) = \frac{1}{4}x + \frac{1}{4}$. For $y'' - 4y' + 4y = -\sin x$ try $y_{p2}(x) = A \cos x + B \sin x$.

Then $y'_{p2} = -A \sin x + B \cos x$ and $y''_{p2} = -A \cos x - B \sin x$. Substituting, we have

$$(-A \cos x - B \sin x) - 4(-A \sin x + B \cos x) + 4(A \cos x + B \sin x) = -\sin x \Rightarrow$$

$$(3A - 4B) \cos x + (4A + 3B) \sin x = -\sin x. \text{ Thus } 3A - 4B = 0 \text{ and } 4A + 3B = -1,$$

giving $A = -\frac{4}{25}$ and $B = -\frac{3}{25}$, so $y_{p2}(x) = -\frac{4}{25} \cos x - \frac{3}{25} \sin x$. The general solution is

$$y(x) = y_c(x) + y_{p1}(x) + y_{p2}(x) = c_1e^{2x} + c_2xe^{2x} + \frac{1}{4}x + \frac{1}{4} - \frac{4}{25} \cos x - \frac{3}{25} \sin x.$$

7. The auxiliary equation is $r^2 - 2r + 5 = 0$ with roots $r = 1 \pm 2i$, so the complementary solution is

$y_c(x) = e^x(c_1 \cos 2x + c_2 \sin 2x)$. Try the particular solution $y_p(x) = A \cos x + B \sin x$, so $y'_p = -A \sin x + B \cos x$ and $y''_p = -A \cos x - B \sin x$. Substituting, we have

$$(-A \cos x - B \sin x) - 2(-A \sin x + B \cos x) + 5(A \cos x + B \sin x) = \sin x \Rightarrow$$

$$(4A - 2B) \cos x + (2A + 4B) \sin x = \sin x. \text{ Then } 4A - 2B = 0, 2A + 4B = 1 \Rightarrow A = \frac{1}{10}, B = \frac{1}{5} \text{ and the general}$$

solution is $y(x) = y_c(x) + y_p(x) = e^x(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{10} \cos x + \frac{1}{5} \sin x$. But $1 = y(0) = c_1 + \frac{1}{10} \Rightarrow c_1 = \frac{9}{10}$

and $1 = y'(0) = 2c_2 + c_1 + \frac{1}{5} \Rightarrow c_2 = -\frac{1}{20}$. Thus the solution to the initial-value problem is

$$y(x) = e^x \left(\frac{9}{10} \cos 2x - \frac{1}{20} \sin 2x \right) + \frac{1}{10} \cos x + \frac{1}{5} \sin x.$$

8. The auxiliary equation is $r^2 - 1 = 0$ with roots $r = \pm 1$, so the complementary solution is $y_c(x) = c_1e^x + c_2e^{-x}$. Try the

particular solution $y_p(x) = (Ax + B)e^{2x}$, so $y'_p = (2Ax + A + 2B)e^{2x}$ and $y''_p = (4Ax + 4A + 4B)e^{2x}$. Substituting, we have $(4Ax + 4A + 4B)e^{2x} - (Ax + B)e^{2x} = xe^{2x} \Rightarrow (3Ax + 4A + 3B)e^{2x} = xe^{2x}$. Then $3A = 1 \Rightarrow A = \frac{1}{3}$ and

$$4A + 3B = 0 \Rightarrow B = -\frac{4}{9}, \text{ and the general solution is } y(x) = y_c(x) + y_p(x) = c_1e^x + c_2e^{-x} + \left(\frac{1}{3}x - \frac{4}{9}\right)e^{2x}. \text{ But}$$

$0 = y(0) = c_1 + c_2 - \frac{4}{9}$ and $1 = y'(0) = c_1 - c_2 - \frac{5}{9} \Rightarrow c_1 = 1, c_2 = -\frac{5}{9}$. Thus the solution to the initial-value

problem is $y(x) = e^x - \frac{5}{9}e^{-x} + \left(\frac{1}{3}x - \frac{4}{9}\right)e^{2x}$.

9. The auxiliary equation is $r^2 - r = 0$ with roots $r = 0, r = 1$ so the complementary solution is $y_c(x) = c_1 + c_2 e^x$.

Try $y_p(x) = x(Ax + B)e^x$ so that no term in y_p is a solution of the complementary equation. Then

$y'_p = (Ax^2 + (2A + B)x + B)e^x$ and $y''_p = (Ax^2 + (4A + B)x + (2A + 2B))e^x$. Substitution into the differential equation gives $(Ax^2 + (4A + B)x + (2A + 2B))e^x - (Ax^2 + (2A + B)x + B)e^x = xe^x \Rightarrow (2Ax + (2A + B))e^x = xe^x \Rightarrow A = \frac{1}{2}, B = -1$. Thus $y_p(x) = (\frac{1}{2}x^2 - x)e^x$ and the general solution is $y(x) = c_1 + c_2 e^x + (\frac{1}{2}x^2 - x)e^x$. But $2 = y(0) = c_1 + c_2$ and $1 = y'(0) = c_2 - 1$, so $c_2 = 2$ and $c_1 = 0$. The solution to the initial-value problem is $y(x) = 2e^x + (\frac{1}{2}x^2 - x)e^x = e^x(\frac{1}{2}x^2 - x + 2)$.

10. $y_c(x) = c_1 e^x + c_2 e^{-2x}$. For $y'' + y' - 2y = x$ try $y_{p1}(x) = Ax + B$. Then $y'_{p1} = A, y''_{p1} = 0$, and substitution gives

$0 + A - 2(Ax + B) = x \Rightarrow A = -\frac{1}{2}, B = -\frac{1}{4}$, so $y_{p1}(x) = -\frac{1}{2}x - \frac{1}{4}$. For $y'' + y' - 2y = \sin 2x$ try

$y_{p2}(x) = A \cos 2x + B \sin 2x$. Then $y'_{p2} = -2A \sin 2x + 2B \cos 2x, y''_{p2} = -4A \cos 2x - 4B \sin 2x$, and substitution

gives $(-4A \cos 2x - 4B \sin 2x) + (-2A \sin 2x + 2B \cos 2x) - 2(A \cos 2x + B \sin 2x) = \sin 2x \Rightarrow A = -\frac{1}{20},$

$B = -\frac{3}{20}$. Thus $y_{p2}(x) = -\frac{1}{20} \cos 2x - \frac{3}{20} \sin 2x$ and the general solution is

$y(x) = c_1 e^x + c_2 e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{1}{20} \cos 2x - \frac{3}{20} \sin 2x$. But $1 = y(0) = c_1 + c_2 - \frac{1}{4} - \frac{1}{20}$ and

$0 = y'(0) = c_1 - 2c_2 - \frac{1}{2} - \frac{3}{10} \Rightarrow c_1 = \frac{17}{15}$ and $c_2 = \frac{1}{6}$. Thus the solution to the initial-value problem is

$y(x) = \frac{17}{15}e^x + \frac{1}{6}e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{1}{20} \cos 2x - \frac{3}{20} \sin 2x$.

11. The auxiliary equation is $r^2 + 3r + 2 = (r + 1)(r + 2) = 0$, so $r = -1, r = -2$ and $y_c(x) = c_1 e^{-x} + c_2 e^{-2x}$.

Try $y_p = A \cos x + B \sin x \Rightarrow y'_p = -A \sin x + B \cos x, y''_p = -A \cos x - B \sin x$. Substituting into the differential equation gives $(-A \cos x - B \sin x) + 3(-A \sin x + B \cos x) + 2(A \cos x + B \sin x) = \cos x$ or

$(A + 3B) \cos x + (-3A + B) \sin x = \cos x$. Then solving the equations

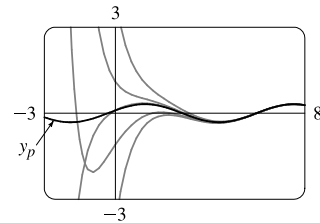
$A + 3B = 1, -3A + B = 0$ gives $A = \frac{1}{10}, B = \frac{3}{10}$ and the general

solution is $y(x) = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{10} \cos x + \frac{3}{10} \sin x$. The graph

shows y_p and several other solutions. Notice that all solutions are

asymptotic to y_p as $x \rightarrow \infty$. Except for y_p , all solutions approach either ∞

or $-\infty$ as $x \rightarrow -\infty$.



12. The auxiliary equation is $r^2 + 4 = 0 \Rightarrow r = \pm 2i$, so $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. Try $y_p = Ae^{-x} \Rightarrow$

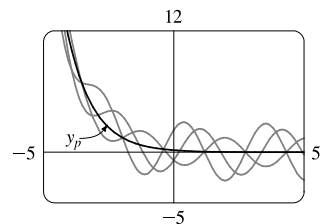
$y'_p = -Ae^{-x}, y''_p = Ae^{-x}$. Substituting into the differential equation gives $Ae^{-x} + 4Ae^{-x} = e^{-x} \Rightarrow$

$5A = 1 \Rightarrow A = \frac{1}{5}$, so $y_p = \frac{1}{5}e^{-x}$ and the general solution is

$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5}e^{-x}$. We graph y_p along with several

other solutions. All of the solutions except y_p oscillate around $y_p = \frac{1}{5}e^{-x}$,

and all solutions approach ∞ as $x \rightarrow -\infty$.



13. Here $y_c(x) = c_1 e^{2x} + c_2 e^{-x}$, and a trial solution is $y_p(x) = (Ax + B)e^x \cos x + (Cx + D)e^x \sin x$.
14. Here $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. For $y'' + 4y = \cos 4x$ try $y_{p1}(x) = A \cos 4x + B \sin 4x$ and for $y'' + 4y = \cos 2x$ try $y_{p2}(x) = x(C \cos 2x + D \sin 2x)$ (so that no term of y_{p2} is a solution of the complementary equation). Thus a trial solution is $y_p(x) = y_{p1}(x) + y_{p2}(x) = A \cos 4x + B \sin 4x + Cx \cos 2x + Dx \sin 2x$.
15. Here $y_c(x) = c_1 e^{2x} + c_2 e^x$. For $y'' - 3y' + 2y = e^x$ try $y_{p1}(x) = Axe^x$ (since $y = Ae^x$ is a solution of the complementary equation) and for $y'' - 3y' + 2y = \sin x$ try $y_{p2}(x) = B \cos x + C \sin x$. Thus a trial solution is $y_p(x) = y_{p1}(x) + y_{p2}(x) = Axe^x + B \cos x + C \sin x$.
16. Since $y_c(x) = c_1 e^x + c_2 e^{-4x}$ try $y_p(x) = x(Ax^3 + Bx^2 + Cx + D)e^x$ so that no term of $y_p(x)$ satisfies the complementary equation.
17. Since $y_c(x) = e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$ we try $y_p(x) = x(Ax^2 + Bx + C)e^{-x} \cos 3x + x(Dx^2 + Ex + F)e^{-x} \sin 3x$ (so that no term of y_p is a solution of the complementary equation).
18. Here $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. For $y'' + 4y = e^{3x}$ try $y_{p1}(x) = Ae^{3x}$ and for $y'' + 4y = x \sin 2x$ try $y_{p2}(x) = x(Bx + C) \cos 2x + x(Dx + E) \sin 2x$ (so that no term of y_{p2} is a solution of the complementary equation).

Note: Solving Equations (7) and (9) in The Method of Variation of Parameters gives

$$u'_1 = -\frac{Gy_2}{a(y_1 y'_2 - y_2 y'_1)} \quad \text{and} \quad u'_2 = \frac{Gy_1}{a(y_1 y'_2 - y_2 y'_1)}$$

We will use these equations rather than resolving the system in each of the remaining exercises in this section.

19. (a) Here $4r^2 + 1 = 0 \Rightarrow r = \pm \frac{1}{2}i$ and $y_c(x) = c_1 \cos(\frac{1}{2}x) + c_2 \sin(\frac{1}{2}x)$. We try a particular solution of the form

$$y_p(x) = A \cos x + B \sin x \Rightarrow y'_p = -A \sin x + B \cos x \text{ and } y''_p = -A \cos x - B \sin x. \text{ Then the equation}$$

$$4y'' + y = \cos x \text{ becomes } 4(-A \cos x - B \sin x) + (A \cos x + B \sin x) = \cos x \text{ or}$$

$$-3A \cos x - 3B \sin x = \cos x \Rightarrow A = -\frac{1}{3}, B = 0. \text{ Thus, } y_p(x) = -\frac{1}{3} \cos x \text{ and the general solution is}$$

$$y(x) = y_c(x) + y_p(x) = c_1 \cos(\frac{1}{2}x) + c_2 \sin(\frac{1}{2}x) - \frac{1}{3} \cos x.$$

- (b) From (a) we know that $y_c(x) = c_1 \cos \frac{x}{2} + c_2 \sin \frac{x}{2}$. Setting $y_1 = \cos \frac{x}{2}$, $y_2 = \sin \frac{x}{2}$, we have

$$y_1 y'_2 - y_2 y'_1 = \frac{1}{2} \cos^2 \frac{x}{2} + \frac{1}{2} \sin^2 \frac{x}{2} = \frac{1}{2}. \text{ Thus } u'_1 = -\frac{\cos x \sin \frac{x}{2}}{4 \cdot \frac{1}{2}} = -\frac{1}{2} \cos(2 \cdot \frac{x}{2}) \sin \frac{x}{2} = -\frac{1}{2} (2 \cos^2 \frac{x}{2} - 1) \sin \frac{x}{2}$$

$$\text{and } u'_2 = \frac{\cos x \cos \frac{x}{2}}{4 \cdot \frac{1}{2}} = \frac{1}{2} \cos(2 \cdot \frac{x}{2}) \cos \frac{x}{2} = \frac{1}{2} (1 - 2 \sin^2 \frac{x}{2}) \cos \frac{x}{2}. \text{ Then}$$

$$u_1(x) = \int (\frac{1}{2} \sin \frac{x}{2} - \cos^2 \frac{x}{2} \sin \frac{x}{2}) dx = -\cos \frac{x}{2} + \frac{2}{3} \cos^3 \frac{x}{2} \text{ and}$$

$$u_2(x) = \int (\frac{1}{2} \cos \frac{x}{2} - \sin^2 \frac{x}{2} \cos \frac{x}{2}) dx = \sin \frac{x}{2} - \frac{2}{3} \sin^3 \frac{x}{2}. \text{ Thus}$$

$$\begin{aligned} y_p(x) &= (-\cos \frac{x}{2} + \frac{2}{3} \cos^3 \frac{x}{2}) \cos \frac{x}{2} + (\sin \frac{x}{2} - \frac{2}{3} \sin^3 \frac{x}{2}) \sin \frac{x}{2} = -(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}) + \frac{2}{3} (\cos^4 \frac{x}{2} - \sin^4 \frac{x}{2}) \\ &= -\cos(2 \cdot \frac{x}{2}) + \frac{2}{3} (\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}) (\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}) = -\cos x + \frac{2}{3} \cos x = -\frac{1}{3} \cos x \end{aligned}$$

$$\text{and the general solution is } y(x) = y_c(x) + y_p(x) = c_1 \cos \frac{x}{2} + c_2 \sin \frac{x}{2} - \frac{1}{3} \cos x.$$

20. (a) Here $r^2 - 2r - 3 = (r - 3)(r + 1) = 0 \Rightarrow r = 3, r = -1$ and the complementary solution is

$$y_c(x) = c_1 e^{3x} + c_2 e^{-x}. \text{ A particular solution is of the form } y_p(x) = Ax + B \Rightarrow y'_p = A, y''_p = 0, \text{ and}$$

$$\text{substituting into the differential equation gives } 0 - 2A - 3(Ax + B) = x + 2 \text{ or } -3Ax + (-2A - 3B) = x + 2,$$

$$\text{so } A = -\frac{1}{3} \text{ and } -2A - 3B = 2 \Rightarrow B = -\frac{4}{9}. \text{ Thus } y_p(x) = -\frac{1}{3}x - \frac{4}{9} \text{ and the general solution is}$$

$$y(x) = y_c(x) + y_p(x) = c_1 e^{3x} + c_2 e^{-x} - \frac{1}{3}x - \frac{4}{9}.$$

- (b) In (a), $y_c(x) = c_1 e^{3x} + c_2 e^{-x}$, so set $y_1 = e^{3x}, y_2 = e^{-x}$. Then $y_1 y'_2 - y_2 y'_1 = -e^{3x} e^{-x} - 3e^{3x} e^{-x} = -4e^{2x}$ so

$$u'_1 = -\frac{(x+2)e^{-x}}{-4e^{2x}} = \frac{1}{4}(x+2)e^{-3x} \Rightarrow u_1(x) = \frac{1}{4} \int (x+2)e^{-3x} dx = \frac{1}{4} \left[-\frac{1}{3}(x+2)e^{-3x} - \frac{1}{9}e^{-3x} \right] \text{ [by parts]}$$

$$\text{and } u'_2 = \frac{(x+2)e^{3x}}{-4e^{2x}} = -\frac{1}{4}(x+2)e^x \Rightarrow u_2(x) = -\frac{1}{4} \int (x+2)e^x dx = -\frac{1}{4}[(x+2)e^x - e^x] \text{ [by parts].}$$

$$\text{Hence } y_p(x) = \frac{1}{4} \left[\left(-\frac{1}{3}x - \frac{7}{9} \right) e^{-3x} \right] e^{3x} - \frac{1}{4}[(x+1)e^x] e^{-x} = -\frac{1}{3}x - \frac{4}{9} \text{ and}$$

$$y(x) = y_c(x) + y_p(x) = c_1 e^{3x} + c_2 e^{-x} - \frac{1}{3}x - \frac{4}{9}.$$

21. (a) $r^2 - 2r + 1 = (r - 1)^2 = 0 \Rightarrow r = 1$, so the complementary solution is $y_c(x) = c_1 e^x + c_2 x e^x$. A particular solution

$$\text{is of the form } y_p(x) = A e^{2x}. \text{ Thus } 4A e^{2x} - 4A e^{2x} + A e^{2x} = e^{2x} \Rightarrow A e^{2x} = e^{2x} \Rightarrow A = 1 \Rightarrow y_p(x) = e^{2x}.$$

$$\text{So a general solution is } y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 x e^x + e^{2x}.$$

- (b) From (a), $y_c(x) = c_1 e^x + c_2 x e^x$, so set $y_1 = e^x, y_2 = x e^x$. Then, $y_1 y'_2 - y_2 y'_1 = e^{2x}(1+x) - x e^{2x} = e^{2x}$ and so

$$u'_1 = -x e^x \Rightarrow u_1(x) = -\int x e^x dx = -(x-1)e^x \text{ [by parts]} \text{ and } u'_2 = e^x \Rightarrow u_2(x) = \int e^x dx = e^x. \text{ Hence}$$

$$y_p(x) = (1-x)e^{2x} + x e^{2x} = e^{2x} \text{ and the general solution is } y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 x e^x + e^{2x}.$$

22. (a) Here $r^2 - r = r(r - 1) = 0 \Rightarrow r = 0, 1$ and $y_c(x) = c_1 + c_2 e^x$ and so we try a particular solution of the form

$$y_p(x) = A x e^x. \text{ Thus, after calculating the necessary derivatives, we get } y'' - y' = e^x \Rightarrow$$

$$A e^x(2+x) - A e^x(1+x) = e^x \Rightarrow A = 1. \text{ Thus } y_p(x) = x e^x \text{ and the general solution is } y(x) = c_1 + c_2 e^x + x e^x.$$

- (b) From (a) we know that $y_c(x) = c_1 + c_2 e^x$, so setting $y_1 = 1, y_2 = e^x$, then $y_1 y'_2 - y_2 y'_1 = e^x - 0 = e^x$. Thus

$$u'_1 = -e^{2x}/e^x = -e^x \text{ and } u'_2 = e^x/e^x = 1. \text{ Then } u_1(x) = -\int e^x dx = -e^x \text{ and } u_2(x) = x. \text{ Thus}$$

$$y_p(x) = -e^x + x e^x \text{ and the general solution is } y(x) = c_1 + c_2 e^x - e^x + x e^x = c_1 + c_3 e^x + x e^x.$$

23. As in Example 5, $y_c(x) = c_1 \sin x + c_2 \cos x$, so set $y_1 = \sin x, y_2 = \cos x$. Then $y_1 y'_2 - y_2 y'_1 = -\sin^2 x - \cos^2 x = -1$,

$$\text{so } u'_1 = -\frac{\sec^2 x \cos x}{-1} = \sec x \Rightarrow u_1(x) = \int \sec x dx = \ln(\sec x + \tan x) \text{ for } 0 < x < \frac{\pi}{2},$$

$$\text{and } u'_2 = \frac{\sec^2 x \sin x}{-1} = -\sec x \tan x \Rightarrow u_2(x) = -\sec x. \text{ Hence}$$

$$y_p(x) = \ln(\sec x + \tan x) \cdot \sin x - \sec x \cdot \cos x = \sin x \ln(\sec x + \tan x) - 1 \text{ and the general solution is}$$

$$y(x) = c_1 \sin x + c_2 \cos x + \sin x \ln(\sec x + \tan x) - 1.$$

24. As in Exercise 23, $y_c(x) = c_1 \sin x + c_2 \cos x$, $y_1 = \sin x$, $y_2 = \cos x$, and $y_1 y'_2 - y_2 y'_1 = -1$. Then

$$u'_1 = -\frac{\sec^3 x \cos x}{-1} = \sec^2 x \Rightarrow u_1(x) = \tan x \text{ and } u'_2 = \frac{\sec^3 x \sin x}{-1} = -\sec^2 x \tan x \Rightarrow$$

$$u_2(x) = -\int \tan x \sec^2 x dx = -\frac{1}{2} \tan^2 x. \text{ Hence}$$

$$y_p(x) = \tan x \sin x - \frac{1}{2} \tan^2 x \cos x = \tan x \sin x - \frac{1}{2} \tan x \sin x = \frac{1}{2} \tan x \sin x \text{ and the general solution}$$

$$\text{is } y(x) = c_1 \sin x + c_2 \cos x + \frac{1}{2} \tan x \sin x.$$

25. $y_1 = e^x$, $y_2 = e^{2x}$ and $y_1 y'_2 - y_2 y'_1 = e^{3x}$. So $u'_1 = \frac{-e^{2x}}{(1+e^{-x})e^{3x}} = -\frac{e^{-x}}{1+e^{-x}}$ and

$$u_1(x) = \int -\frac{e^{-x}}{1+e^{-x}} dx = \ln(1+e^{-x}). \quad u'_2 = \frac{e^x}{(1+e^{-x})e^{3x}} = \frac{e^x}{e^{3x}+e^{2x}} \text{ so}$$

$$u_2(x) = \int \frac{e^x}{e^{3x}+e^{2x}} dx = \ln\left(\frac{e^x+1}{e^x}\right) - e^{-x} = \ln(1+e^{-x}) - e^{-x}. \text{ Hence}$$

$$y_p(x) = e^x \ln(1+e^{-x}) + e^{2x}[\ln(1+e^{-x}) - e^{-x}] \text{ and the general solution is}$$

$$y(x) = [c_1 + \ln(1+e^{-x})]e^x + [c_2 - e^{-x} + \ln(1+e^{-x})]e^{2x}.$$

26. $y_1 = e^{-x}$, $y_2 = e^{-2x}$ and $y_1 y'_2 - y_2 y'_1 = -e^{-3x}$. So $u'_1 = -\frac{(\sin e^x)e^{-2x}}{-e^{-3x}} = e^x \sin e^x$

$$\text{and } u'_2 = \frac{(\sin e^x)e^{-x}}{-e^{-3x}} = -e^{2x} \sin e^x. \text{ Hence } u_1(x) = \int e^x \sin e^x dx = -\cos e^x \text{ and}$$

$$u_2(x) = \int -e^{2x} \sin e^x dx = e^x \cos e^x - \sin e^x. \text{ Then } y_p(x) = -e^{-x} \cos e^x - e^{-2x}[\sin e^x - e^x \cos e^x]$$

$$\text{and the general solution is } y(x) = (c_1 - \cos e^x)e^{-x} + [c_2 - \sin e^x + e^x \cos e^x]e^{-2x}.$$

27. $r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow r = 1$ so $y_c(x) = c_1 e^x + c_2 x e^x$. Thus $y_1 = e^x$, $y_2 = x e^x$ and

$$y_1 y'_2 - y_2 y'_1 = e^x(x+1)e^x - x e^x e^x = e^{2x}. \text{ So } u'_1 = -\frac{x e^x \cdot e^x / (1+x^2)}{e^{2x}} = -\frac{x}{1+x^2} \Rightarrow$$

$$u_1 = -\int \frac{x}{1+x^2} dx = -\frac{1}{2} \ln(1+x^2), \quad u'_2 = \frac{e^x \cdot e^x / (1+x^2)}{e^{2x}} = \frac{1}{1+x^2} \Rightarrow u_2 = \int \frac{1}{1+x^2} dx = \tan^{-1} x \text{ and}$$

$$y_p(x) = -\frac{1}{2} e^x \ln(1+x^2) + x e^x \tan^{-1} x. \text{ Hence the general solution is } y(x) = e^x [c_1 + c_2 x - \frac{1}{2} \ln(1+x^2) + x \tan^{-1} x].$$

28. $y_1 = e^{-2x}$, $y_2 = x e^{-2x}$ and $y_1 y'_2 - y_2 y'_1 = e^{-4x}$. Then $u'_1 = \frac{-e^{-2x} x e^{-2x}}{x^3 e^{-4x}} = -\frac{1}{x^2}$ so $u_1(x) = x^{-1}$ and

$$u'_2 = \frac{e^{-2x} e^{-2x}}{x^3 e^{-4x}} = \frac{1}{x^3} \text{ so } u_2(x) = -\frac{1}{2x^2}. \text{ Thus } y_p(x) = \frac{e^{-2x}}{x} - \frac{x e^{-2x}}{2x^2} = \frac{e^{-2x}}{2x} \text{ and the general solution is}$$

$$y(x) = e^{-2x} [c_1 + c_2 x + 1/(2x)].$$

17.3 Applications of Second-Order Differential Equations

1. By Hooke's Law $k(0.25) = 25$ so $k = 100$ is the spring constant and the differential equation is $5x'' + 100x = 0$.

The auxiliary equation is $5r^2 + 100 = 0$ with roots $r = \pm 2\sqrt{5}i$, so the general solution to the differential equation is

$$x(t) = c_1 \cos(2\sqrt{5}t) + c_2 \sin(2\sqrt{5}t). \text{ We are given that } x(0) = 0.35 \Rightarrow c_1 = 0.35 \text{ and } x'(0) = 0 \Rightarrow$$

$$2\sqrt{5}c_2 = 0 \Rightarrow c_2 = 0, \text{ so the position of the mass after } t \text{ seconds is } x(t) = 0.35 \cos(2\sqrt{5}t).$$

2. By Hooke's Law $k(0.4) = 32$ so $k = \frac{32}{0.4} = 80$ is the spring constant and the differential equation is $8x'' + 80x = 0$.

The general solution is $x(t) = c_1 \cos(\sqrt{10}t) + c_2 \sin(\sqrt{10}t)$. But $0 = x(0) = c_1$ and $1 = x'(0) = \sqrt{10}c_2 \Rightarrow$

$$c_2 = \frac{1}{\sqrt{10}}, \text{ so the position of the mass after } t \text{ seconds is } x(t) = \frac{1}{\sqrt{10}} \sin(\sqrt{10}t).$$

3. $k(0.5) = 6$ or $k = 12$ is the spring constant, so the initial-value problem is $2x'' + 14x' + 12x = 0$, $x(0) = 1$, $x'(0) = 0$.

The general solution is $x(t) = c_1 e^{-6t} + c_2 e^{-t}$. But $1 = x(0) = c_1 + c_2$ and $0 = x'(0) = -6c_1 - c_2$. Thus the position is given by $x(t) = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}$.

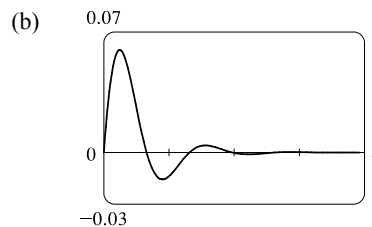
4. (a) $k(0.25) = 13 \Rightarrow k = 52$, so the differential equation is

$$2x'' + 8x' + 52x = 0 \text{ with general solution}$$

$$x(t) = e^{-2t} [c_1 \cos(\sqrt{22}t) + c_2 \sin(\sqrt{22}t)]. \text{ Then } 0 = x(0) = c_1$$

$$\text{and } 0.5 = x'(0) = \sqrt{22}c_2 \Rightarrow c_2 = \frac{1}{2\sqrt{22}}, \text{ so the position is}$$

$$\text{given by } x(t) = \frac{1}{2\sqrt{22}} e^{-2t} \sin(\sqrt{22}t).$$



5. For critical damping we need $c^2 - 4mk = 0$ or $m = c^2/(4k) = 14^2/(4 \cdot 12) = \frac{49}{12}$ kg.

6. For critical damping we need $c^2 = 4mk$ or $c = 2\sqrt{mk} = 2\sqrt{2 \cdot 52} = 4\sqrt{26}$.

7. We are given $m = 1$, $k = 100$, $x(0) = -0.1$ and $x'(0) = 0$. From (3), the differential equation is $\frac{d^2x}{dt^2} + c \frac{dx}{dt} + 100x = 0$ with auxiliary equation $r^2 + cr + 100 = 0$.

If $c = 10$, we have two complex roots $r = -5 \pm 5\sqrt{3}i$, so the motion is underdamped and the solution is

$$x = e^{-5t} [c_1 \cos(5\sqrt{3}t) + c_2 \sin(5\sqrt{3}t)]. \text{ Then } -0.1 = x(0) = c_1 \text{ and } 0 = x'(0) = 5\sqrt{3}c_2 - 5c_1 \Rightarrow c_2 = -\frac{1}{10\sqrt{3}},$$

$$\text{so } x = e^{-5t} \left[-0.1 \cos(5\sqrt{3}t) - \frac{1}{10\sqrt{3}} \sin(5\sqrt{3}t) \right].$$

If $c = 15$, we again have underdamping since the auxiliary equation has roots $r = -\frac{15}{2} \pm \frac{5\sqrt{7}}{2}i$. The general solution is

$$x = e^{-15t/2} \left[c_1 \cos\left(\frac{5\sqrt{7}}{2}t\right) + c_2 \sin\left(\frac{5\sqrt{7}}{2}t\right) \right], \text{ so } -0.1 = x(0) = c_1 \text{ and } 0 = x'(0) = \frac{5\sqrt{7}}{2}c_2 - \frac{15}{2}c_1 \Rightarrow c_2 = -\frac{3}{10\sqrt{7}}.$$

$$\text{Thus } x = e^{-15t/2} \left[-0.1 \cos\left(\frac{5\sqrt{7}}{2}t\right) - \frac{3}{10\sqrt{7}} \sin\left(\frac{5\sqrt{7}}{2}t\right) \right].$$

For $c = 20$, we have equal roots $r_1 = r_2 = -10$, so the oscillation is critically damped and the solution is

$$x = (c_1 + c_2 t)e^{-10t}. \text{ Then } -0.1 = x(0) = c_1 \text{ and } 0 = x'(0) = -10c_1 + c_2 \Rightarrow c_2 = -1, \text{ so } x = (-0.1 - t)e^{-10t}.$$

If $c = 25$ the auxiliary equation has roots $r_1 = -5, r_2 = -20$, so we have overdamping and the solution is

$$x = c_1 e^{-5t} + c_2 e^{-20t}. \text{ Then } -0.1 = x(0) = c_1 + c_2 \text{ and } 0 = x'(0) = -5c_1 - 20c_2 \Rightarrow c_1 = -\frac{2}{15} \text{ and } c_2 = \frac{1}{30},$$

$$\text{so } x = -\frac{2}{15}e^{-5t} + \frac{1}{30}e^{-20t}.$$

If $c = 30$ we have roots $r = -15 \pm 5\sqrt{5}$, so the motion is

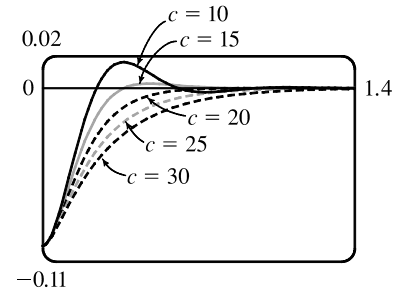
$$\text{overdamped and the solution is } x = c_1 e^{(-15+5\sqrt{5})t} + c_2 e^{(-15-5\sqrt{5})t}.$$

$$\text{Then } -0.1 = x(0) = c_1 + c_2 \text{ and}$$

$$0 = x'(0) = (-15 + 5\sqrt{5})c_1 + (-15 - 5\sqrt{5})c_2 \Rightarrow$$

$$c_1 = \frac{-5-3\sqrt{5}}{100} \text{ and } c_2 = \frac{-5+3\sqrt{5}}{100}, \text{ so}$$

$$x = \left(\frac{-5-3\sqrt{5}}{100}\right)e^{(-15+5\sqrt{5})t} + \left(\frac{-5+3\sqrt{5}}{100}\right)e^{(-15-5\sqrt{5})t}.$$



8. We are given $m = 1, c = 10, x(0) = 0$ and $x'(0) = 1$. The differential equation is $\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + kx = 0$ with auxiliary

equation $r^2 + 10r + k = 0$. $k = 10$: the auxiliary equation has roots $r = -5 \pm \sqrt{15}$ so we have overdamping and the

solution is $x = c_1 e^{(-5+\sqrt{15})t} + c_2 e^{(-5-\sqrt{15})t}$. Entering the initial conditions gives $c_1 = \frac{1}{2\sqrt{15}}$ and $c_2 = -\frac{1}{2\sqrt{15}}$, so

$$x = \frac{1}{2\sqrt{15}}e^{(-5+\sqrt{15})t} - \frac{1}{2\sqrt{15}}e^{(-5-\sqrt{15})t}.$$

$k = 20$: $r = -5 \pm \sqrt{5}$ and the solution is $x = c_1 e^{(-5+\sqrt{5})t} + c_2 e^{(-5-\sqrt{5})t}$ so again the motion is overdamped.

The initial conditions give $c_1 = \frac{1}{2\sqrt{5}}$ and $c_2 = -\frac{1}{2\sqrt{5}}$, so $x = \frac{1}{2\sqrt{5}}e^{(-5+\sqrt{5})t} - \frac{1}{2\sqrt{5}}e^{(-5-\sqrt{5})t}$.

$k = 25$: we have equal roots $r_1 = r_2 = -5$, so the motion is critically damped and the solution is $x = (c_1 + c_2 t)e^{-5t}$.

The initial conditions give $c_1 = 0$ and $c_2 = 1$, so $x = te^{-5t}$.

$k = 30$: $r = -5 \pm \sqrt{5}i$ so the motion is underdamped and the solution is $x = e^{-5t}[c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)]$.

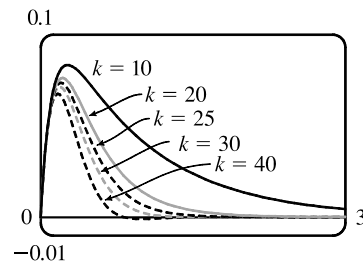
The initial conditions give $c_1 = 0$ and $c_2 = \frac{1}{\sqrt{5}}$, so $x = \frac{1}{\sqrt{5}}e^{-5t} \sin(\sqrt{5}t)$.

$k = 40$: $r = -5 \pm \sqrt{15}i$ so we again have underdamping.

The solution is $x = e^{-5t}[c_1 \cos(\sqrt{15}t) + c_2 \sin(\sqrt{15}t)]$,

and the initial conditions give $c_1 = 0$ and $c_2 = \frac{1}{\sqrt{15}}$.

$$\text{Thus } x = \frac{1}{\sqrt{15}}e^{-5t} \sin(\sqrt{15}t).$$



9. The differential equation is $mx'' + kx = F_0 \cos \omega_0 t$ and $\omega_0 \neq \omega = \sqrt{k/m}$. Here the auxiliary equation is $mr^2 + k = 0$

with roots $\pm \sqrt{k/m}i = \pm \omega i$ so $x_c(t) = c_1 \cos \omega t + c_2 \sin \omega t$. Since $\omega_0 \neq \omega$, try $x_p(t) = A \cos \omega_0 t + B \sin \omega_0 t$.

Then we need $(m)(-\omega_0^2)(A \cos \omega_0 t + B \sin \omega_0 t) + k(A \cos \omega_0 t + B \sin \omega_0 t) = F_0 \cos \omega_0 t$ or $A(k - m\omega_0^2) = F_0$ and

$B(k - m\omega_0^2) = 0$. Hence $B = 0$ and $A = \frac{F_0}{k - m\omega_0^2} = \frac{F_0}{m(\omega^2 - \omega_0^2)}$ since $\omega^2 = \frac{k}{m}$. Thus the motion of the mass is given

$$\text{by } x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t.$$

10. As in Exercise 9, $x_c(t) = c_1 \cos \omega t + c_2 \sin \omega t$. But the natural frequency of the system equals the frequency of the external force, so try $x_p(t) = t(A \cos \omega t + B \sin \omega t)$. Then we need

$$m(2\omega B - \omega^2 At) \cos \omega t - m(2\omega A + \omega^2 Bt) \sin \omega t + kAt \cos \omega t + kBt \sin \omega t = F_0 \cos \omega t \text{ or } 2m\omega B = F_0 \text{ and } -2m\omega A = 0 \text{ [noting } -m\omega^2 A + kA = 0 \text{ and } -m\omega^2 B + kB = 0 \text{ since } \omega^2 = k/m]. \text{ Hence the general solution is } x(t) = c_1 \cos \omega t + c_2 \sin \omega t + [F_0 t / (2m\omega)] \sin \omega t.$$

11. From Equation 6, $x(t) = f(t) + g(t)$ where $f(t) = c_1 \cos \omega t + c_2 \sin \omega t$ and $g(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t$. Then f

is periodic, with period $\frac{2\pi}{\omega}$, and if $\omega \neq \omega_0$, g is periodic with period $\frac{2\pi}{\omega_0}$. If $\frac{\omega}{\omega_0}$ is a rational number, then we can say

$$\frac{\omega}{\omega_0} = \frac{a}{b} \Rightarrow a = \frac{b\omega}{\omega_0} \text{ where } a \text{ and } b \text{ are non-zero integers. Then}$$

$$x\left(t + a \cdot \frac{2\pi}{\omega}\right) = f\left(t + a \cdot \frac{2\pi}{\omega}\right) + g\left(t + a \cdot \frac{2\pi}{\omega}\right) = f(t) + g\left(t + \frac{b\omega}{\omega_0} \cdot \frac{2\pi}{\omega}\right) = f(t) + g\left(t + b \cdot \frac{2\pi}{\omega_0}\right) = f(t) + g(t) = x(t)$$

so $x(t)$ is periodic.

12. (a) The graph of $x = c_1 e^{rt} + c_2 t e^{rt}$ has a t -intercept when $c_1 e^{rt} + c_2 t e^{rt} = 0 \Leftrightarrow e^{rt}(c_1 + c_2 t) = 0 \Leftrightarrow c_1 = -c_2 t$.

Since $t > 0$, x has a t -intercept if and only if c_1 and c_2 have opposite signs.

- (b) For $t > 0$, the graph of x crosses the t -axis when $c_1 e^{r_1 t} + c_2 e^{r_2 t} = 0 \Leftrightarrow c_2 e^{r_2 t} = -c_1 e^{r_1 t} \Leftrightarrow$

$$c_2 = -c_1 \frac{e^{r_1 t}}{e^{r_2 t}} = -c_1 e^{(r_1 - r_2)t}. \text{ But } r_1 > r_2 \Rightarrow r_1 - r_2 > 0 \text{ and since } t > 0, e^{(r_1 - r_2)t} > 1. \text{ Thus}$$

$$|c_2| = |c_1| e^{(r_1 - r_2)t} > |c_1|, \text{ and the graph of } x \text{ can cross the } t\text{-axis only if } |c_2| > |c_1|.$$

13. Here the initial-value problem for the charge is $Q'' + 20Q' + 500Q = 12$, $Q(0) = Q'(0) = 0$. Then

$$Q_c(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t) \text{ and try } Q_p(t) = A \Rightarrow 500A = 12 \text{ or } A = \frac{3}{125}.$$

The general solution is $Q(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t) + \frac{3}{125}$. But $0 = Q(0) = c_1 + \frac{3}{125}$ and

$$Q'(t) = I(t) = e^{-10t}[(-10c_1 + 20c_2) \cos 20t + (-10c_2 - 20c_1) \sin 20t] \text{ but } 0 = Q'(0) = -10c_1 + 20c_2. \text{ Thus the charge}$$

$$\text{is } Q(t) = -\frac{1}{250} e^{-10t}(6 \cos 20t + 3 \sin 20t) + \frac{3}{125} \text{ and the current is } I(t) = e^{-10t}\left(\frac{3}{5}\right) \sin 20t.$$

14. (a) Here the initial-value problem for the charge is $2Q'' + 24Q' + 200Q = 12$ with $Q(0) = 0.001$ and $Q'(0) = 0$.

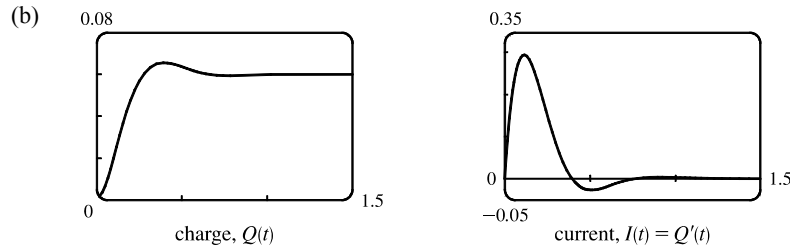
$$\text{Then } Q_c(t) = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) \text{ and try } Q_p(t) = A \Rightarrow A = \frac{3}{50} \text{ and the general solution is}$$

$$Q(t) = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) + \frac{3}{50}. \text{ But } 0.001 = Q(0) = c_1 + \frac{3}{50} \text{ so } c_1 = -0.059. \text{ Also}$$

$$Q'(t) = I(t) = e^{-6t}[(-6c_1 + 8c_2) \cos 8t + (-6c_2 - 8c_1) \sin 8t] \text{ and } 0 = Q'(0) = -6c_1 + 8c_2 \text{ so}$$

$$c_2 = -0.04425. \text{ Hence the charge is } Q(t) = -e^{-6t}(0.059 \cos 8t + 0.04425 \sin 8t) + \frac{3}{50} \text{ and the current is}$$

$$I(t) = e^{-6t}(0.7375) \sin 8t.$$



15. As in Exercise 13, $Q_c(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t)$ but $E(t) = 12 \sin 10t$ so try

$Q_p(t) = A \cos 10t + B \sin 10t$. Substituting into the differential equation gives

$$(-100A + 200B + 500A) \cos 10t + (-100B - 200A + 500B) \sin 10t = 12 \sin 10t \Rightarrow$$

$400A + 200B = 0$ and $400B - 200A = 12$. Thus $A = -\frac{3}{250}$, $B = \frac{3}{125}$ and the general solution is

$$Q(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t) - \frac{3}{250} \cos 10t + \frac{3}{125} \sin 10t. \text{ But } 0 = Q(0) = c_1 - \frac{3}{250} \text{ so } c_1 = \frac{3}{250}.$$

Also $Q'(t) = \frac{3}{25} \sin 10t + \frac{6}{25} \cos 10t + e^{-10t}[(-10c_1 + 20c_2) \cos 20t + (-10c_2 - 20c_1) \sin 20t]$ and

$0 = Q'(0) = \frac{6}{25} - 10c_1 + 20c_2$ so $c_2 = -\frac{3}{500}$. Hence the charge is given by

$$Q(t) = e^{-10t} \left[\frac{3}{250} \cos 20t - \frac{3}{500} \sin 20t \right] - \frac{3}{250} \cos 10t + \frac{3}{125} \sin 10t.$$

16. (a) As in Exercise 14, $Q_c(t) = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t)$ but try $Q_p(t) = A \cos 10t + B \sin 10t$. Substituting into the

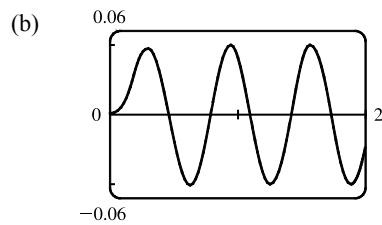
differential equation gives $(-200A + 240B + 200A) \cos 10t + (-200B - 240A + 200B) \sin 10t = 12 \sin 10t$,

so $B = 0$ and $A = -\frac{1}{20}$. Hence, the general solution is $Q(t) = e^{-6t}(c_1 \cos 8t + c_2 \sin 8t) - \frac{1}{20} \cos 10t$. But

$$0.001 = Q(0) = c_1 - \frac{1}{20}, \quad Q'(t) = e^{-6t}[(-6c_1 + 8c_2) \cos 8t + (-6c_2 - 8c_1) \sin 8t] - \frac{1}{2} \sin 10t \text{ and}$$

$0 = Q'(0) = -6c_1 + 8c_2$, so $c_1 = 0.051$ and $c_2 = 0.03825$. Thus the charge is given by

$$Q(t) = e^{-6t}(0.051 \cos 8t + 0.03825 \sin 8t) - \frac{1}{20} \cos 10t.$$



17. $x(t) = A \cos(\omega t + \delta) \Leftrightarrow x(t) = A[\cos \omega t \cos \delta - \sin \omega t \sin \delta] \Leftrightarrow x(t) = A\left(\frac{c_1}{A} \cos \omega t + \frac{c_2}{A} \sin \omega t\right)$ where

$$\cos \delta = c_1/A \text{ and } \sin \delta = -c_2/A \Leftrightarrow x(t) = c_1 \cos \omega t + c_2 \sin \omega t. \text{ [Note that } \cos^2 \delta + \sin^2 \delta = 1 \Rightarrow c_1^2 + c_2^2 = A^2.]$$

18. (a) We approximate $\sin \theta$ by θ and, with $L = 1$ and $g = 9.8$, the differential equation becomes $\frac{d^2 \theta}{dt^2} + 9.8\theta = 0$. The auxiliary

equation is $r^2 + 9.8 = 0 \Rightarrow r = \pm \sqrt{9.8}i$, so the general solution is $\theta(t) = c_1 \cos(\sqrt{9.8}t) + c_2 \sin(\sqrt{9.8}t)$.

Then $0.2 = \theta(0) = c_1$ and $1 = \theta'(0) = \sqrt{9.8} c_2 \Rightarrow c_2 = \frac{1}{\sqrt{9.8}}$, so the equation is

$$\theta(t) = 0.2 \cos(\sqrt{9.8} t) + \frac{1}{\sqrt{9.8}} \sin(\sqrt{9.8} t).$$

(b) $\theta'(t) = -0.2 \sqrt{9.8} \sin(\sqrt{9.8} t) + \cos(\sqrt{9.8} t) = 0$ or $\tan(\sqrt{9.8} t) = \frac{5}{\sqrt{9.8}}$, so the critical numbers are

$$t = \frac{1}{\sqrt{9.8}} \tan^{-1}\left(\frac{5}{\sqrt{9.8}}\right) + \frac{n}{\sqrt{9.8}} \pi \quad (n \text{ any integer}). \text{ The maximum angle from the vertical is}$$

$$\theta\left(\frac{1}{\sqrt{9.8}} \tan^{-1}\left(\frac{5}{\sqrt{9.8}}\right)\right) \approx 0.377 \text{ radians (or about } 21.7^\circ).$$

(c) From part (b), the critical numbers of $\theta(t)$ are spaced $\frac{\pi}{\sqrt{9.8}}$ apart, and the time between successive maximum values

$$\text{is } 2\left(\frac{\pi}{\sqrt{9.8}}\right). \text{ Thus the period of the pendulum is } \frac{2\pi}{\sqrt{9.8}} \approx 2.007 \text{ seconds.}$$

(d) $\theta(t) = 0 \Rightarrow 0.2 \cos(\sqrt{9.8} t) + \frac{1}{\sqrt{9.8}} \sin(\sqrt{9.8} t) = 0 \Rightarrow \tan(\sqrt{9.8} t) = -0.2 \sqrt{9.8} \Rightarrow$

$$t = \frac{1}{\sqrt{9.8}} [\tan^{-1}(-0.2 \sqrt{9.8}) + \pi] \approx 0.825 \text{ seconds.}$$

(e) $\theta'(0.825) \approx -1.180 \text{ rad/s.}$

17.4 Series Solutions

1. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and the given equation, $y' - y = 0$, becomes

$$\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0. \text{ Replacing } n \text{ by } n+1 \text{ in the first sum gives } \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=0}^{\infty} c_n x^n = 0, \text{ so}$$

$$\sum_{n=0}^{\infty} [(n+1) c_{n+1} - c_n] x^n = 0. \text{ Equating coefficients gives } (n+1) c_{n+1} - c_n = 0, \text{ so the recursion relation is}$$

$$c_{n+1} = \frac{c_n}{n+1}, \quad n = 0, 1, 2, \dots. \text{ Then } c_1 = c_0, c_2 = \frac{1}{2} c_1 = \frac{c_0}{2}, c_3 = \frac{1}{3} c_2 = \frac{1}{3} \cdot \frac{1}{2} c_0 = \frac{c_0}{3!}, c_4 = \frac{1}{4} c_3 = \frac{c_0}{4!}, \text{ and}$$

$$\text{in general, } c_n = \frac{c_0}{n!}. \text{ Thus, the solution is } y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{c_0}{n!} x^n = c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = c_0 e^x.$$

2. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = xy \Rightarrow y' - xy = 0 \Rightarrow \sum_{n=1}^{\infty} n c_n x^{n-1} - x \sum_{n=0}^{\infty} c_n x^n = 0$ or

$$\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0. \text{ Replacing } n \text{ with } n+1 \text{ in the first sum and } n \text{ with } n-1 \text{ in the second}$$

$$\text{gives } \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=1}^{\infty} c_{n-1} x^n = 0 \text{ or } c_1 + \sum_{n=1}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=1}^{\infty} c_{n-1} x^n = 0. \text{ Thus,}$$

$$c_1 + \sum_{n=1}^{\infty} [(n+1) c_{n+1} - c_{n-1}] x^n = 0. \text{ Equating coefficients gives } c_1 = 0 \text{ and } (n+1) c_{n+1} - c_{n-1} = 0. \text{ Thus, the}$$

$$\text{recursion relation is } c_{n+1} = \frac{c_{n-1}}{n+1}, \quad n = 1, 2, \dots. \text{ But } c_1 = 0, \text{ so } c_3 = 0 \text{ and } c_5 = 0 \text{ and in general } c_{2n+1} = 0. \text{ Also,}$$

$c_2 = \frac{c_0}{2}$, $c_4 = \frac{c_2}{4} = \frac{c_0}{4 \cdot 2} = \frac{c_0}{2^2 \cdot 2!}$, $c_6 = \frac{c_4}{6} = \frac{c_0}{6 \cdot 4 \cdot 2} = \frac{c_0}{2^3 \cdot 3!}$ and in general $c_{2n} = \frac{c_0}{2^n \cdot n!}$. Thus, the solution

$$\text{is } y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} = \sum_{n=0}^{\infty} \frac{c_0}{2^n \cdot n!} x^{2n} = c_0 \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = c_0 e^{x^2/2}.$$

3. Assuming $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$ and

$$-x^2 y = - \sum_{n=0}^{\infty} c_n x^{n+2} = - \sum_{n=2}^{\infty} c_{n-2} x^n. \text{ Hence, the equation } y' = x^2 y \text{ becomes } \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=2}^{\infty} c_{n-2} x^n = 0$$

$$\text{or } c_1 + 2c_2 x + \sum_{n=2}^{\infty} [(n+1)c_{n+1} - c_{n-2}] x^n = 0. \text{ Equating coefficients gives } c_1 = c_2 = 0 \text{ and } c_{n+1} = \frac{c_{n-2}}{n+1}$$

for $n = 2, 3, \dots$. But $c_1 = 0$, so $c_4 = 0$ and $c_7 = 0$ and in general $c_{3n+1} = 0$. Similarly $c_2 = 0$ so $c_{3n+2} = 0$. Finally

$$c_3 = \frac{c_0}{3}, c_6 = \frac{c_3}{6} = \frac{c_0}{6 \cdot 3} = \frac{c_0}{3^2 \cdot 2!}, c_9 = \frac{c_6}{9} = \frac{c_0}{9 \cdot 6 \cdot 3} = \frac{c_0}{3^3 \cdot 3!}, \dots, \text{ and } c_{3n} = \frac{c_0}{3^n \cdot n!}. \text{ Thus, the solution}$$

$$\text{is } y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{3n} x^{3n} = \sum_{n=0}^{\infty} \frac{c_0}{3^n \cdot n!} x^{3n} = c_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = c_0 \sum_{n=0}^{\infty} \frac{(x^3/3)^n}{n!} = c_0 e^{x^3/3}.$$

4. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$. Then the differential equation becomes

$$(x-3) \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + 2 \sum_{n=0}^{\infty} c_n x^n = 0 \Rightarrow \sum_{n=0}^{\infty} (n+1) c_{n+1} x^{n+1} - 3 \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + 2 \sum_{n=0}^{\infty} c_n x^n = 0 \Rightarrow$$

$$\sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} 3(n+1) c_{n+1} x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0 \Rightarrow \sum_{n=0}^{\infty} [(n+2)c_n - 3(n+1)c_{n+1}] x^n = 0$$

[since $\sum_{n=1}^{\infty} n c_n x^n = \sum_{n=0}^{\infty} n c_n x^n$]. Equating coefficients gives $(n+2)c_n - 3(n+1)c_{n+1} = 0$, thus the recursion relation is

$$c_{n+1} = \frac{(n+2)c_n}{3(n+1)}, n = 0, 1, 2, \dots. \text{ Then } c_1 = \frac{2c_0}{3}, c_2 = \frac{3c_1}{3(2)} = \frac{3c_0}{3^2}, c_3 = \frac{4c_2}{3(3)} = \frac{4c_0}{3^3}, c_4 = \frac{5c_3}{3(4)} = \frac{5c_0}{3^4}, \text{ and}$$

$$\text{in general, } c_n = \frac{(n+1)c_0}{3^n}. \text{ Thus the solution is } y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n.$$

$$\left[\text{Note that } c_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n = \frac{9c_0}{(3-x)^2} \text{ for } |x| < 3. \right]$$

5. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$. The differential equation

$$\text{becomes } \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0 \text{ or } \sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} + n c_n + c_n] x^n = 0$$

[since $\sum_{n=1}^{\infty} n c_n x^n = \sum_{n=0}^{\infty} n c_n x^n$]. Equating coefficients gives $(n+2)(n+1) c_{n+2} + (n+1) c_n = 0$, thus the recursion

$$\text{relation is } c_{n+2} = \frac{-(n+1)c_n}{(n+2)(n+1)} = -\frac{c_n}{n+2}, n = 0, 1, 2, \dots. \text{ Then the even}$$

$$\text{coefficients are given by } c_2 = -\frac{c_0}{2}, c_4 = -\frac{c_2}{4} = \frac{c_0}{2 \cdot 4}, c_6 = -\frac{c_4}{6} = -\frac{c_0}{2 \cdot 4 \cdot 6}, \text{ and in general,}$$

$c_{2n} = (-1)^n \frac{c_0}{2 \cdot 4 \cdot \dots \cdot 2n} = \frac{(-1)^n c_0}{2^n n!}$. The odd coefficients are $c_3 = -\frac{c_1}{3}$, $c_5 = -\frac{c_3}{5} = \frac{c_1}{3 \cdot 5}$, $c_7 = -\frac{c_5}{7} = -\frac{c_1}{3 \cdot 5 \cdot 7}$,

and in general, $c_{2n+1} = (-1)^n \frac{c_1}{3 \cdot 5 \cdot 7 \cdot \dots \cdot (2n+1)} = \frac{(-2)^n n! c_1}{(2n+1)!}$. The solution is

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n+1)!} x^{2n+1}.$$

6. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$. Hence, the equation $y'' = y$

becomes $\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - \sum_{n=0}^{\infty} c_n x^n = 0$ or $\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - c_n] x^n = 0$. So the recursion relation

is $c_{n+2} = \frac{c_n}{(n+2)(n+1)}$, $n = 0, 1, \dots$. Given c_0 and c_1 , $c_2 = \frac{c_0}{2 \cdot 1}$, $c_4 = \frac{c_2}{4 \cdot 3} = \frac{c_0}{4!}$, $c_6 = \frac{c_4}{6 \cdot 5} = \frac{c_0}{6!}$, \dots ,

$c_{2n} = \frac{c_0}{(2n)!}$ and $c_3 = \frac{c_1}{3 \cdot 2}$, $c_5 = \frac{c_3}{5 \cdot 4} = \frac{c_1}{5 \cdot 4 \cdot 3 \cdot 2} = \frac{c_1}{5!}$, $c_7 = \frac{c_5}{7 \cdot 6} = \frac{c_1}{7!}$, \dots , $c_{2n+1} = \frac{c_1}{(2n+1)!}$. Thus, the solution

is $y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} + \sum_{n=0}^{\infty} c_{2n+1} x^{2n+1} = c_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$. The solution can be written

as $y(x) = c_0 \cosh x + c_1 \sinh x$ [or $y(x) = c_0 \frac{e^x + e^{-x}}{2} + c_1 \frac{e^x - e^{-x}}{2} = \frac{c_0 + c_1}{2} e^x + \frac{c_0 - c_1}{2} e^{-x}$].

7. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$ and $y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$. Then

$$(x-1)y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^{n+1} - \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n = \sum_{n=1}^{\infty} n(n+1) c_{n+1} x^n - \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n.$$

Since $\sum_{n=1}^{\infty} n(n+1) c_{n+1} x^n = \sum_{n=0}^{\infty} n(n+1) c_{n+1} x^n$, the differential equation becomes

$$\sum_{n=0}^{\infty} n(n+1) c_{n+1} x^n - \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty} [n(n+1) c_{n+1} - (n+2)(n+1) c_{n+2} + (n+1) c_{n+1}] x^n = 0 \text{ or } \sum_{n=0}^{\infty} [(n+1)^2 c_{n+1} - (n+2)(n+1) c_{n+2}] x^n = 0.$$

Equating coefficients gives $(n+1)^2 c_{n+1} - (n+2)(n+1) c_{n+2} = 0$ for $n = 0, 1, 2, \dots$. Then the recursion relation is

$c_{n+2} = \frac{(n+1)^2}{(n+2)(n+1)} c_{n+1} = \frac{n+1}{n+2} c_{n+1}$, so given c_0 and c_1 , we have $c_2 = \frac{1}{2} c_1$, $c_3 = \frac{2}{3} c_2 = \frac{1}{3} c_1$, $c_4 = \frac{3}{4} c_3 = \frac{1}{4} c_1$, and

in general $c_n = \frac{c_1}{n}$, $n = 1, 2, 3, \dots$. Thus the solution is $y(x) = c_0 + c_1 \sum_{n=1}^{\infty} \frac{x^n}{n}$. Note that the solution can be expressed as

$c_0 - c_1 \ln(1-x)$ for $|x| < 1$.

8. Assuming $y(x) = \sum_{n=0}^{\infty} c_n x^n$, $y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$ and

$-xy(x) = -\sum_{n=0}^{\infty} c_n x^{n+1} = -\sum_{n=1}^{\infty} c_{n-1} x^n$. The equation $y'' = xy$ becomes

$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty} c_{n-1}x^n = 0$ or $2c_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} - c_{n-1}]x^n = 0$. Equating coefficients

gives $c_2 = 0$ and $c_{n+2} = \frac{c_{n-1}}{(n+2)(n+1)}$ for $n = 1, 2, \dots$. Since $c_2 = 0$, $c_{3n+2} = 0$ for $n = 0, 1, 2, \dots$. Given c_0 ,

$$c_3 = \frac{c_0}{3 \cdot 2}, c_6 = \frac{c_3}{6 \cdot 5} = \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}, \dots, c_{3n} = \frac{c_0}{3n(3n-1)(3n-3)(3n-4) \cdots 6 \cdot 5 \cdot 3 \cdot 2}.$$
 Given c_1 , $c_4 = \frac{c_1}{4 \cdot 3}$,

$$c_7 = \frac{c_4}{7 \cdot 6} = \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}, \dots, c_{3n+1} = \frac{c_1}{(3n+1)3n(3n-2)(3n-3) \cdots 7 \cdot 6 \cdot 4 \cdot 3}.$$
 The solution can be written

$$\text{as } y(x) = c_0 \sum_{n=0}^{\infty} \frac{(3n-2)(3n-5) \cdots 7 \cdot 4 \cdot 1}{(3n)!} x^{3n} + c_1 \sum_{n=0}^{\infty} \frac{(3n-1)(3n-4) \cdots 8 \cdot 5 \cdot 2}{(3n+1)!} x^{3n+1}.$$

9. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $-xy'(x) = -x \sum_{n=1}^{\infty} n c_n x^{n-1} = - \sum_{n=1}^{\infty} n c_n x^n = - \sum_{n=0}^{\infty} n c_n x^n$,

$$y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n, \text{ and the equation } y'' - xy' - y = 0 \text{ becomes}$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - n c_n - c_n]x^n = 0. \text{ Thus, the recursion relation is}$$

$$c_{n+2} = \frac{n c_n + c_n}{(n+2)(n+1)} = \frac{c_n(n+1)}{(n+2)(n+1)} = \frac{c_n}{n+2} \text{ for } n = 0, 1, 2, \dots. \text{ One of the given conditions is } y(0) = 1. \text{ But}$$

$$y(0) = \sum_{n=0}^{\infty} c_n(0)^n = c_0 + 0 + 0 + \cdots = c_0, \text{ so } c_0 = 1. \text{ Hence, } c_2 = \frac{c_0}{2} = \frac{1}{2}, c_4 = \frac{c_2}{4} = \frac{1}{2 \cdot 4}, c_6 = \frac{c_4}{6} = \frac{1}{2 \cdot 4 \cdot 6}, \dots,$$

$$c_{2n} = \frac{1}{2^n n!}. \text{ The other given condition is } y'(0) = 0. \text{ But } y'(0) = \sum_{n=1}^{\infty} n c_n(0)^{n-1} = c_1 + 0 + 0 + \cdots = c_1, \text{ so } c_1 = 0.$$

By the recursion relation, $c_3 = \frac{c_1}{3} = 0$, $c_5 = 0$, \dots , $c_{2n+1} = 0$ for $n = 0, 1, 2, \dots$. Thus, the solution to the initial-value

$$\text{problem is } y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = e^{x^2/2}.$$

10. Assuming that $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $x^2 y = \sum_{n=0}^{\infty} c_n x^{n+2}$ and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=-2}^{\infty} (n+4)(n+3)c_{n+4}x^{n+2} = 2c_2 + 6c_3x + \sum_{n=0}^{\infty} (n+4)(n+3)c_{n+4}x^{n+2}.$$

Thus, the equation $y'' + x^2 y = 0$ becomes $2c_2 + 6c_3x + \sum_{n=0}^{\infty} [(n+4)(n+3)c_{n+4} + c_n]x^{n+2} = 0$. So $c_2 = c_3 = 0$ and

the recursion relation is $c_{n+4} = -\frac{c_n}{(n+4)(n+3)}$, $n = 0, 1, 2, \dots$. But $c_1 = y'(0) = 0 = c_2 = c_3$ and by the recursion

$$\text{relation, } c_{4n+1} = c_{4n+2} = c_{4n+3} = 0 \text{ for } n = 0, 1, 2, \dots. \text{ Also, } c_0 = y(0) = 1, \text{ so } c_4 = -\frac{c_0}{4 \cdot 3} = -\frac{1}{4 \cdot 3},$$

$$c_8 = -\frac{c_4}{8 \cdot 7} = \frac{(-1)^2}{8 \cdot 7 \cdot 4 \cdot 3}, \dots, c_{4n} = \frac{(-1)^n}{4n(4n-1)(4n-4)(4n-5) \cdots 4 \cdot 3}.$$
 Thus, the solution to the initial-value

$$\text{problem is } y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + \sum_{n=0}^{\infty} c_{4n} x^{4n} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{x^{4n}}{4n(4n-1)(4n-4)(4n-5) \cdots 4 \cdot 3}.$$

11. Assuming that $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $xy = x \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+1}$, $x^2 y' = x^2 \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} n c_n x^{n+1}$,

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=-1}^{\infty} (n+3)(n+2)c_{n+3} x^{n+1} \quad [\text{replace } n \text{ with } n+3]$$

$$= 2c_2 + \sum_{n=0}^{\infty} (n+3)(n+2)c_{n+3} x^{n+1},$$

and the equation $y'' + x^2 y' + xy = 0$ becomes $2c_2 + \sum_{n=0}^{\infty} [(n+3)(n+2)c_{n+3} + n c_n + c_n] x^{n+1} = 0$. So $c_2 = 0$ and the

recursion relation is $c_{n+3} = \frac{-n c_n - c_n}{(n+3)(n+2)} = -\frac{(n+1)c_n}{(n+3)(n+2)}$, $n = 0, 1, 2, \dots$. But $c_0 = y(0) = 0 = c_2$ and by the

recursion relation, $c_{3n} = c_{3n+2} = 0$ for $n = 0, 1, 2, \dots$. Also, $c_1 = y'(0) = 1$, so $c_4 = -\frac{2c_1}{4 \cdot 3} = -\frac{2}{4 \cdot 3}$,

$c_7 = -\frac{5c_4}{7 \cdot 6} = (-1)^2 \frac{2 \cdot 5}{7 \cdot 6 \cdot 4 \cdot 3} = (-1)^2 \frac{2^2 5^2}{7!}$, \dots , $c_{3n+1} = (-1)^n \frac{2^2 5^2 \cdots (3n-1)^2}{(3n+1)!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = x + \sum_{n=1}^{\infty} \left[(-1)^n \frac{2^2 5^2 \cdots (3n-1)^2 x^{3n+1}}{(3n+1)!} \right].$$

12. (a) Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $x^2 y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^{n+2}$,

$$xy'(x) = \sum_{n=1}^{\infty} n c_n x^n = \sum_{n=-1}^{\infty} (n+2)c_{n+2} x^{n+2} = c_1 x + \sum_{n=0}^{\infty} (n+2)c_{n+2} x^{n+2}, \text{ and the equation}$$

$x^2 y'' + xy' + x^2 y = 0$ becomes $c_1 x + \sum_{n=0}^{\infty} \{[(n+2)(n+1) + (n+2)]c_{n+2} + c_n\} x^{n+2} = 0$. So $c_1 = 0$ and the

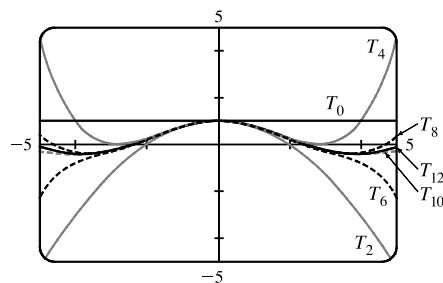
recursion relation is $c_{n+2} = -\frac{c_n}{(n+2)^2}$, $n = 0, 1, 2, \dots$. But $c_1 = y'(0) = 0$ so $c_{2n+1} = 0$ for $n = 0, 1, 2, \dots$.

Also, $c_0 = y(0) = 1$, so $c_2 = -\frac{1}{2^2}$, $c_4 = -\frac{c_2}{4^2} = (-1)^2 \frac{1}{4^2 2^2} = (-1)^2 \frac{1}{2^4 (2!)^2}$, $c_6 = -\frac{c_4}{6^2} = (-1)^3 \frac{1}{2^6 (3!)^2}$, \dots ,

$c_{2n} = (-1)^n \frac{1}{2^{2n} (n!)^2}$. The solution is $y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} (n!)^2}$.

(b) The Taylor polynomials T_0 to T_{12} are shown in the graph.

Because T_{10} and T_{12} are close together throughout the interval $[-5, 5]$, it is reasonable to assume that T_{12} is a good approximation to the Bessel function on that interval.



17 Review

TRUE-FALSE QUIZ

1. True. See Theorem 17.1.3.
2. False. The differential equation is not homogeneous.
3. True. $\cosh x$ and $\sinh x$ are linearly independent solutions of this linear homogeneous equation.
4. False. $y = Ae^x$ is a solution of the complementary equation, so we have to take $y_p(x) = Axe^x$.

EXERCISES

1. The auxiliary equation is $4r^2 - 1 = 0 \Rightarrow (2r + 1)(2r - 1) = 0 \Rightarrow r = \pm \frac{1}{2}$. Then the general solution is $y = c_1 e^{x/2} + c_2 e^{-x/2}$.
2. The auxiliary equation is $r^2 - 2r + 10 = 0 \Rightarrow r = 1 \pm 3i$, so $y = e^x(c_1 \cos 3x + c_2 \sin 3x)$.
3. The auxiliary equation is $r^2 + 3 = 0 \Rightarrow r = \pm \sqrt{3}i$. Then the general solution is $y = c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)$.
4. The auxiliary equation is $r^2 + 8r + 16 = 0 \Rightarrow (r + 4)^2 = 0 \Rightarrow r = -4$, so the general solution is $y = c_1 e^{-4x} + c_2 x e^{-4x}$.
5. $r^2 - 4r + 5 = 0 \Rightarrow r = 2 \pm i$, so $y_c(x) = e^{2x}(c_1 \cos x + c_2 \sin x)$. Try $y_p(x) = Ae^{2x} \Rightarrow y'_p = 2Ae^{2x}$ and $y''_p = 4Ae^{2x}$. Substitution into the differential equation gives $4Ae^{2x} - 8Ae^{2x} + 5Ae^{2x} = e^{2x} \Rightarrow A = 1$ and the general solution is $y(x) = e^{2x}(c_1 \cos x + c_2 \sin x) + e^{2x}$.
6. $r^2 + r - 2 = 0 \Rightarrow r = 1, r = -2$ and $y_c(x) = c_1 e^x + c_2 e^{-2x}$. Try $y_p(x) = Ax^2 + Bx + C \Rightarrow y'_p = 2Ax + B$ and $y''_p = 2A$. Substitution gives $2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2 \Rightarrow A = B = -\frac{1}{2}, C = -\frac{3}{4}$ so the general solution is $y(x) = c_1 e^x + c_2 e^{-2x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$.
7. $r^2 - 2r + 1 = 0 \Rightarrow r = 1$ and $y_c(x) = c_1 e^x + c_2 x e^x$. Try $y_p(x) = (Ax + B) \cos x + (Cx + D) \sin x \Rightarrow y'_p = (C - Ax - B) \sin x + (A + Cx + D) \cos x$ and $y''_p = (2C - B - Ax) \cos x + (-2A - D - Cx) \sin x$. Substitution gives $(-2Cx + 2C - 2A - 2D) \cos x + (2Ax - 2A + 2B - 2C) \sin x = x \cos x \Rightarrow A = 0, B = C = D = -\frac{1}{2}$. The general solution is $y(x) = c_1 e^x + c_2 x e^x - \frac{1}{2} \cos x - \frac{1}{2}(x + 1) \sin x$.
8. $r^2 + 4 = 0 \Rightarrow r = \pm 2i$ and $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$. Try $y_p(x) = Ax \cos 2x + Bx \sin 2x$ so that no term of y_p is a solution of the complementary equation. Then $y'_p = (A + 2Bx) \cos 2x + (B - 2Ax) \sin 2x$ and $y''_p = (4B - 4Ax) \cos 2x + (-4A - 4Bx) \sin 2x$. Substitution gives $4B \cos 2x - 4A \sin 2x = \sin 2x \Rightarrow A = -\frac{1}{4}$ and $B = 0$. The general solution is $y(x) = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}x \cos 2x$.

9. $r^2 - r - 6 = 0 \Rightarrow r = -2, r = 3$ and $y_c(x) = c_1 e^{-2x} + c_2 e^{3x}$. For $y'' - y' - 6y = 1$, try $y_{p1}(x) = A$. Then $y'_{p1}(x) = y''_{p1}(x) = 0$ and substitution into the differential equation gives $A = -\frac{1}{6}$. For $y'' - y' - 6y = e^{-2x}$ try $y_{p2}(x) = Bx e^{-2x}$ [since $y = B e^{-2x}$ satisfies the complementary equation]. Then $y'_{p2} = (B - 2Bx)e^{-2x}$ and $y''_{p2} = (4Bx - 4B)e^{-2x}$, and substitution gives $-5B e^{-2x} = e^{-2x} \Rightarrow B = -\frac{1}{5}$. The general solution then is $y(x) = c_1 e^{-2x} + c_2 e^{3x} + y_{p1}(x) + y_{p2}(x) = c_1 e^{-2x} + c_2 e^{3x} - \frac{1}{6} - \frac{1}{5} x e^{-2x}$.
10. Using variation of parameters, $y_c(x) = c_1 \cos x + c_2 \sin x$, $u'_1(x) = -\csc x \sin x = -1 \Rightarrow u_1(x) = -x$, and $u'_2(x) = \frac{\csc x \cos x}{x} = \cot x \Rightarrow u_2(x) = \ln |\sin x| \Rightarrow y_p = -x \cos x + \sin x \ln |\sin x|$. The solution is $y(x) = (c_1 - x) \cos x + (c_2 + \ln |\sin x|) \sin x$.
11. The auxiliary equation is $r^2 + 6r = 0$ and the general solution is $y(x) = c_1 + c_2 e^{-6x} = k_1 + k_2 e^{-6(x-1)}$. But $3 = y(1) = k_1 + k_2$ and $12 = y'(1) = -6k_2$. Thus $k_2 = -2$, $k_1 = 5$ and the solution is $y(x) = 5 - 2e^{-6(x-1)}$.
12. The auxiliary equation is $r^2 - 6r + 25 = 0$ and the general solution is $y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$. But $2 = y(0) = c_1$ and $1 = y'(0) = 3c_1 + 4c_2$. Thus the solution is $y(x) = e^{3x}(2 \cos 4x - \frac{5}{4} \sin 4x)$.
13. The auxiliary equation is $r^2 - 5r + 4 = 0$ and the general solution is $y(x) = c_1 e^x + c_2 e^{4x}$. But $0 = y(0) = c_1 + c_2$ and $1 = y'(0) = c_1 + 4c_2$, so the solution is $y(x) = \frac{1}{3}(e^{4x} - e^x)$.
14. $y_c(x) = c_1 \cos(x/3) + c_2 \sin(x/3)$. For $9y'' + y = 3x$, try $y_{p1}(x) = Ax + B$. Then $y_{p1}(x) = 3x$. For $9y'' + y = e^{-x}$, try $y_{p2}(x) = A e^{-x}$. Then $9A e^{-x} + A e^{-x} = e^{-x}$ or $y_{p2}(x) = \frac{1}{10} e^{-x}$. Thus the general solution is $y(x) = c_1 \cos(x/3) + c_2 \sin(x/3) + 3x + \frac{1}{10} e^{-x}$. But $1 = y(0) = c_1 + \frac{1}{10}$ and $2 = y'(0) = \frac{1}{3} c_2 + 3 - \frac{1}{10}$, so $c_1 = \frac{9}{10}$ and $c_2 = -\frac{27}{10}$. Hence the solution is $y(x) = \frac{1}{10}[9 \cos(x/3) - 27 \sin(x/3)] + 3x + \frac{1}{10} e^{-x}$.
15. $r^2 + 4r + 29 = 0 \Rightarrow r = -2 \pm 5i$ and the general solution is $y = e^{-2x}(c_1 \cos 5x + c_2 \sin 5x)$. But $1 = y(0) = c_1$ and $-1 = y(\pi) = -c_1 e^{-2\pi} \Rightarrow c_1 = e^{2\pi}$, so there is no solution.
16. $r^2 + 4r + 29 = 0 \Rightarrow r = -2 \pm 5i$ and the general solution is $y = e^{-2x}(c_1 \cos 5x + c_2 \sin 5x)$. But $1 = y(0) = c_1$ and $-e^{-2\pi} = y(\pi) = -c_1 e^{-2\pi} \Rightarrow c_1 = 1$, so c_2 can vary and the solution of the boundary-value problem is $y = e^{-2x}(\cos 5x + c \sin 5x)$, where c is any constant.
17. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$ and the differential equation becomes $\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + (n+1)c_n] x^n = 0$. Thus the recursion relation is $c_{n+2} = -c_n/(n+2)$ for $n = 0, 1, 2, \dots$. But $c_0 = y(0) = 0$, so $c_{2n} = 0$ for $n = 0, 1, 2, \dots$. Also $c_1 = y'(0) = 1$, so $c_3 = -\frac{1}{3}$, $c_5 = \frac{(-1)^2}{3 \cdot 5}$,

$c_7 = \frac{(-1)^3}{3 \cdot 5 \cdot 7} = \frac{(-1)^3 2^3 3!}{7!}, \dots, c_{2n+1} = \frac{(-1)^n 2^n n!}{(2n+1)!}$ for $n = 0, 1, 2, \dots$. Thus the solution to the initial-value problem

$$\text{is } y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!}{(2n+1)!} x^{2n+1}.$$

18. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$ and the differential equation

becomes $\sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} - (n+2) c_n] x^n = 0$. Thus the recursion relation is $c_{n+2} = \frac{c_n}{n+1}$ for

$n = 0, 1, 2, \dots$. Given c_0 and c_1 , we have $c_2 = \frac{c_0}{1}, c_4 = \frac{c_2}{3} = \frac{c_0}{1 \cdot 3}, c_6 = \frac{c_4}{5} = \frac{c_0}{1 \cdot 3 \cdot 5}, \dots$,

$$c_{2n} = \frac{c_0}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = c_0 \frac{2^{n-1}(n-1)!}{(2n-1)!}. \text{ Similarly } c_3 = \frac{c_1}{2}, c_5 = \frac{c_3}{4} = \frac{c_1}{2 \cdot 4},$$

$c_7 = \frac{c_5}{6} = \frac{c_1}{2 \cdot 4 \cdot 6}, \dots, c_{2n+1} = \frac{c_1}{2 \cdot 4 \cdot 6 \cdots 2n} = \frac{c_1}{2^n n!}$. Thus the general solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_0 \sum_{n=1}^{\infty} \frac{2^{n-1}(n-1)! x^{2n}}{(2n-1)!} + c_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^n n!}. \text{ But } \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^n n!} = x \sum_{n=0}^{\infty} \frac{(\frac{1}{2}x^2)^n}{n!} = x e^{x^2/2},$$

$$\text{so } y(x) = c_1 x e^{x^2/2} + c_0 + c_0 \sum_{n=1}^{\infty} \frac{2^{n-1}(n-1)! x^{2n}}{(2n-1)!}.$$

19. Here the initial-value problem is $2Q'' + 40Q' + 400Q = 12$, $Q(0) = 0.01$, $Q'(0) = 0$. Then

$Q_c(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t)$ and we try $Q_p(t) = A$. Thus the general solution is

$$Q(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t) + \frac{3}{100}. \text{ But } 0.01 = Q'(0) = c_1 + 0.03 \text{ and } 0 = Q''(0) = -10c_1 + 10c_2,$$

so $c_1 = -0.02 = c_2$. Hence the charge is given by $Q(t) = -0.02e^{-10t}(\cos 10t + \sin 10t) + 0.03$.

20. By Hooke's Law the spring constant is $k = 64$ and the initial-value problem is $2x'' + 16x' + 64x = 0$, $x(0) = 0$,

$x'(0) = 2.4$. Thus the general solution is $x(t) = e^{-4t}(c_1 \cos 4t + c_2 \sin 4t)$. But $0 = x(0) = c_1$ and

$2.4 = x'(0) = -4c_1 + 4c_2 \Rightarrow c_1 = 0, c_2 = 0.6$. Thus the position of the mass is given by $x(t) = 0.6e^{-4t} \sin 4t$.

21. (a) Since we are assuming that the earth is a solid sphere of uniform density, we can calculate the density ρ as follows:

$$\rho = \frac{\text{mass of earth}}{\text{volume of earth}} = \frac{M}{\frac{4}{3}\pi R^3}. \text{ If } V_r \text{ is the volume of the portion of the earth which lies within a distance } r \text{ of the}$$

$$\text{center, then } V_r = \frac{4}{3}\pi r^3 \text{ and } M_r = \rho V_r = \frac{Mr^3}{R^3}. \text{ Thus } F_r = -\frac{GM_r m}{r^2} = -\frac{GMm}{R^3} r.$$

(b) The particle is acted upon by a varying gravitational force during its motion. By Newton's Second Law of Motion,

$$m \frac{d^2 y}{dt^2} = F_y = -\frac{GMm}{R^3} y, \text{ so } y''(t) = -k^2 y(t) \text{ where } k^2 = \frac{GM}{R^3}. \text{ At the surface, } -mg = F_R = -\frac{GMm}{R^2}, \text{ so}$$

$$g = \frac{GM}{R^2}. \text{ Therefore } k^2 = \frac{g}{R}.$$

- (c) The differential equation $y'' + k^2 y = 0$ has auxiliary equation $r^2 + k^2 = 0$. (This is the r of Section 17.1, not the r measuring distance from the earth's center.) The roots of the auxiliary equation are $\pm ik$, so by (11) in Section 17.1, the general solution of our differential equation for t is $y(t) = c_1 \cos kt + c_2 \sin kt$. It follows that $y'(t) = -c_1 k \sin kt + c_2 k \cos kt$. Now $y(0) = R$ and $y'(0) = 0$, so $c_1 = R$ and $c_2 k = 0$. Thus $y(t) = R \cos kt$ and $y'(t) = -kR \sin kt$. This is simple harmonic motion (see Section 17.3) with amplitude R , frequency k , and phase angle 0. The period is $T = 2\pi/k$. $R \approx 3960$ mi $= 3960 \cdot 5280$ ft and $g = 32$ ft/s², so $k = \sqrt{g/R} \approx 1.24 \times 10^{-3}$ s⁻¹ and $T = 2\pi/k \approx 5079$ s ≈ 85 min.
- (d) $y(t) = 0 \Leftrightarrow \cos kt = 0 \Leftrightarrow kt = \frac{\pi}{2} + \pi n$ for some integer $n \Rightarrow y'(t) = -kR \sin(\frac{\pi}{2} + \pi n) = \pm kR$. Thus the particle passes through the center of the earth with speed $kR \approx 4.899$ mi/s $\approx 17,600$ mi/h.

