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7 ☐ TECHNIQUES OF INTEGRATION

7.1 Integration by Parts

1. Let $u = x$, $dv = e^{2x}\,dx \;\Rightarrow\; du = dx$, $v = \frac{1}{2}e^{2x}$. Then by Equation 2,

$\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C.$

2. Let $u = \ln x$, $dv = \sqrt{x}\,dx \;\Rightarrow\; du = \dfrac{1}{x}\,dx$, $v = \frac{2}{3}x^{3/2}$. Then by Equation 2,

$$\int \sqrt{x}\ln x\,dx = \tfrac{2}{3}x^{3/2}\ln x - \int \tfrac{2}{3}x^{3/2}\cdot\frac{1}{x}\,dx = \tfrac{2}{3}x^{3/2}\ln x - \int \tfrac{2}{3}x^{1/2}\,dx = \tfrac{2}{3}x^{3/2}\ln x - \tfrac{4}{9}x^{3/2} + C.$$

3. Let $u = x$, $dv = \cos 4x\,dx \;\Rightarrow\; du = dx$, $v = \frac{1}{4}\sin 4x$. Then by Equation 2,

$\int x\cos 4x\,dx = \frac{1}{4}x\sin 4x - \int \frac{1}{4}\sin 4x\,dx = \frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + C.$

4. Let $u = \sin^{-1}x$, $dv = dx \;\Rightarrow\; du = \dfrac{1}{\sqrt{1-x^2}}\,dx$, $v = x$. Then by Equation 2,

$$\int \sin^{-1}x\,dx = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1}x - \int \frac{1}{\sqrt{t}}\left(-\frac{1}{2}\,dt\right) \qquad \begin{bmatrix} t = 1 - x^2, \\ dt = -2x\,dx \end{bmatrix}$$

$$= x\sin^{-1}x + \tfrac{1}{2}\int t^{-1/2}\,dt = x\sin^{-1}x + \tfrac{1}{2}\cdot 2t^{1/2} + C = x\sin^{-1}x + \sqrt{1-x^2} + C$$

Note: A mnemonic device which is helpful for selecting u when using integration by parts is the LIATE principle of precedence for u:

Logarithmic
Inverse trigonometric
Algebraic
Trigonometric
Exponential

If the integrand has several factors, then we try to choose among them a u which appears as high as possible on the list. For example, in $\int xe^{2x}\,dx$ the integrand is xe^{2x}, which is the product of an algebraic function (x) and an exponential function (e^{2x}). Since Algebraic appears before Exponential, we choose $u = x$. Sometimes the integration turns out to be similar regardless of the selection of u and dv, but it is advisable to refer to LIATE when in doubt.

5. Let $u = t$, $dv = e^{2t}\,dt \;\Rightarrow\; du = dt$, $v = \frac{1}{2}e^{2t}$. Then by Equation 2,

$\int te^{2t}\,dt = \frac{1}{2}te^{2t} - \int \frac{1}{2}e^{2t}\,dt = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C.$

6. Let $u = y$, $dv = e^{-y}\,dy \;\Rightarrow\; du = dy$, $v = -e^{-y}$. Then by Equation 2,

$\int ye^{-y}\,dy = -ye^{-y} - \int -e^{-y}\,dy = -ye^{-y} + \int e^{-y}\,dy = -ye^{-y} - e^{-y} + C.$

7. Let $u = x$, $dv = \sin 10x\,dx \;\Rightarrow\; du = dx$, $v = -\frac{1}{10}\cos 10x$. Then by Equation 2,

$$\int x\sin 10x\,dx = -\tfrac{1}{10}x\cos 10x - \int -\tfrac{1}{10}\cos 10x\,dx = -\tfrac{1}{10}x\cos 10x + \tfrac{1}{10}\int \cos 10x\,dx$$

$$= -\tfrac{1}{10}x\cos 10x + \tfrac{1}{100}\sin 10x + C$$

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8. Let $u = \pi - x$, $dv = \cos \pi x\, dx \Rightarrow du = -dx$, $v = \dfrac{1}{\pi} \sin \pi x$. Then by Equation 2,

$$\int (\pi - x)\cos \pi x\, dx = \frac{1}{\pi}(\pi - x)\sin \pi x - \int -\frac{1}{\pi}\sin \pi x\, dx = \frac{1}{\pi}(\pi - x)\sin \pi x - \frac{1}{\pi^2}\cos \pi x + C.$$

9. Let $u = \ln w$, $dv = w\, dw \Rightarrow du = \dfrac{1}{w}\, dw$, $v = \dfrac{1}{2}w^2$. Then by Equation 2,

$$\int w \ln w\, dw = \frac{1}{2}w^2 \ln w - \int \frac{1}{2}w^2 \cdot \frac{1}{w}\, dw = \frac{1}{2}w^2 \ln w - \frac{1}{2}\int w\, dw = \frac{1}{2}w^2 \ln w - \frac{1}{4}w^2 + C.$$

10. Let $u = \ln x$, $dv = \dfrac{1}{x^2}\, dx = x^{-2}\, dx \Rightarrow du = \dfrac{1}{x}\, dx = x^{-1}\, dx$, $v = -x^{-1}$. Then by Equation 2,

$$\int \frac{\ln x}{x^2}\, dx = -\frac{\ln x}{x} - \int -x^{-1} \cdot x^{-1}\, dx = -\frac{\ln x}{x} + \int x^{-2}\, dx = -\frac{\ln x}{x} - x^{-1} + C = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

11. First let $u = x^2 + 2x$, $dv = \cos x\, dx \Rightarrow du = (2x + 2)\, dx$, $v = \sin x$. Then by Equation 2, $I = \int (x^2 + 2x)\cos x\, dx = (x^2 + 2x)\sin x - \int (2x + 2)\sin x\, dx$. Next let $U = 2x + 2$, $dV = \sin x\, dx \Rightarrow$ $dU = 2\, dx$, $V = -\cos x$, so $\int (2x + 2)\sin x\, dx = -(2x + 2)\cos x - \int -2\cos x\, dx = -(2x + 2)\cos x + 2\sin x$. Thus, $I = (x^2 + 2x)\sin x + (2x + 2)\cos x - 2\sin x + C$.

12. First let $u = t^2$, $dv = \sin \beta t\, dt \Rightarrow du = 2t\, dt$, $v = -\frac{1}{\beta}\cos \beta t$. Then by Equation 2, $I = \int t^2 \sin \beta t\, dt = -\frac{1}{\beta}t^2 \cos \beta t - \int -\frac{2}{\beta}t \cos \beta t\, dt$. Next let $U = t$, $dV = \cos \beta t\, dt \Rightarrow dU = dt$, $V = \dfrac{1}{\beta}\sin \beta t$, so $\int t \cos \beta t\, dt = \dfrac{1}{\beta}t \sin \beta t - \displaystyle\int \frac{1}{\beta}\sin \beta t\, dt = \frac{1}{\beta}t \sin \beta t + \frac{1}{\beta^2}\cos \beta t$. Thus,

$$I = -\frac{1}{\beta}t^2 \cos \beta t + \frac{2}{\beta}\left(\frac{1}{\beta}t \sin \beta t + \frac{1}{\beta^2}\cos \beta t\right) + C = -\frac{1}{\beta}t^2 \cos \beta t + \frac{2}{\beta^2}t \sin \beta t + \frac{2}{\beta^3}\cos \beta t + C.$$

13. Let $u = \cos^{-1} x$, $dv = dx \Rightarrow du = \dfrac{-1}{\sqrt{1 - x^2}}\, dx$, $v = x$. Then by Equation 2,

$$\begin{aligned} \int \cos^{-1} x\, dx &= x \cos^{-1} x - \int \frac{-x}{\sqrt{1 - x^2}}\, dx = x \cos^{-1} x - \int \frac{1}{\sqrt{t}}\left(\frac{1}{2}\, dt\right) \qquad \begin{bmatrix} t = 1 - x^2, \\ dt = -2x\, dx \end{bmatrix} \\ &= x \cos^{-1} x - \tfrac{1}{2} \cdot 2t^{1/2} + C = x \cos^{-1} x - \sqrt{1 - x^2} + C \end{aligned}$$

14. Let $u = \ln \sqrt{x}$, $dv = dx \Rightarrow du = \dfrac{1}{\sqrt{x}} \cdot \dfrac{1}{2\sqrt{x}}\, dx = \dfrac{1}{2x}\, dx$, $v = x$. Then by Equation 2,

$$\int \ln \sqrt{x}\, dx = x \ln \sqrt{x} - \int x \cdot \frac{1}{2x}\, dx = x \ln \sqrt{x} - \int \frac{1}{2}\, dx = x \ln \sqrt{x} - \frac{1}{2}x + C.$$

Note: We could start by using $\ln \sqrt{x} = \frac{1}{2}\ln x$.

15. Let $u = \ln t$, $dv = t^4\, dt \Rightarrow du = \dfrac{1}{t}\, dt$, $v = \dfrac{1}{5}t^5$. Then by Equation 2,

$$\int t^4 \ln t\, dt = \frac{1}{5}t^5 \ln t - \int \frac{1}{5}t^5 \cdot \frac{1}{t}\, dt = \frac{1}{5}t^5 \ln t - \int \frac{1}{5}t^4\, dt = \frac{1}{5}t^5 \ln t - \frac{1}{25}t^5 + C.$$

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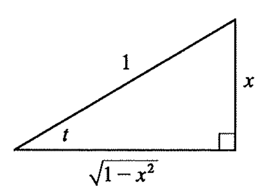
23. First let $u = \sin 3\theta$, $dv = e^{2\theta}\,d\theta \;\Rightarrow\; du = 3\cos 3\theta\,d\theta$, $v = \frac{1}{2}e^{2\theta}$. Then
$I = \int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{2}\int e^{2\theta}\cos 3\theta\,d\theta$. Next let $U = \cos 3\theta$, $dV = e^{2\theta}\,d\theta \;\Rightarrow\; dU = -3\sin 3\theta\,d\theta$,
$V = \frac{1}{2}e^{2\theta}$ to get $\int e^{2\theta}\cos 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\cos 3\theta + \frac{3}{2}\int e^{2\theta}\sin 3\theta\,d\theta$. Substituting in the previous formula gives
$I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}\int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}I \;\Rightarrow$
$\frac{13}{4}I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta + C_1$. Hence, $I = \frac{1}{13}e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$, where $C = \frac{4}{13}C_1$.

24. First let $u = e^{-\theta}$, $dv = \cos 2\theta\,d\theta \;\Rightarrow\; du = -e^{-\theta}\,d\theta$, $v = \frac{1}{2}\sin 2\theta$. Then
$I = \int e^{-\theta}\cos 2\theta\,d\theta = \frac{1}{2}e^{-\theta}\sin 2\theta - \int \frac{1}{2}\sin 2\theta\left(-e^{-\theta}\,d\theta\right) = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\int e^{-\theta}\sin 2\theta\,d\theta$.
Next let $U = e^{-\theta}$, $dV = \sin 2\theta\,d\theta \;\Rightarrow\; dU = -e^{-\theta}\,d\theta$, $V = -\frac{1}{2}\cos 2\theta$, so
$\int e^{-\theta}\sin 2\theta\,d\theta = -\frac{1}{2}e^{-\theta}\cos 2\theta - \int\left(-\frac{1}{2}\right)\cos 2\theta(-e^{-\theta}\,d\theta) = -\frac{1}{2}e^{-\theta}\cos 2\theta - \frac{1}{2}\int e^{-\theta}\cos 2\theta\,d\theta$.
So $I = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\left[\left(-\frac{1}{2}e^{-\theta}\cos 2\theta\right) - \frac{1}{2}I\right] = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta - \frac{1}{4}I \;\Rightarrow$
$\frac{5}{4}I = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1 \;\Rightarrow\; I = \frac{4}{5}\left(\frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1\right) = \frac{2}{5}e^{-\theta}\sin 2\theta - \frac{1}{5}e^{-\theta}\cos 2\theta + C$.

25. First let $u = z^3$, $dv = e^z dz \;\Rightarrow\; du = 3z^2 dz$, $v = e^z$. Then $I_1 = \int z^3e^z dz = z^3e^z - 3\int z^2e^z dz$. Next let $u_1 = z^2$,
$dv_1 = e^z dz \;\Rightarrow\; du_1 = 2z\,dz$, $v_1 = e^z$. Then $I_2 = z^2e^z - 2\int ze^z dz$. Finally, let $u_2 = z$, $dv_2 = e^z dz \;\Rightarrow\; du_2 = dz$,
$v_2 = e^z$. Then $\int ze^z dz = ze^z - \int e^z dz = ze^z - e^z + C_1$. Substituting in the expression for I_2, we get
$I_2 = z^2e^z - 2(ze^z - e^z + C_1) = z^2e^z - 2ze^z + 2e^z - 2C_1$. Substituting the last expression for I_2 into I_1 gives
$I_1 = z^3e^z - 3(z^2e^z - 2ze^z + 2e^z - 2C_1) = z^3e^z - 3z^2e^z + 6ze^z - 6e^z + C$, where $C = 6C_1$.

26. First let $u = (\arcsin x)^2$, $dv = dx \;\Rightarrow\; du = 2\arcsin x \cdot \dfrac{1}{\sqrt{1-x^2}}\,dx$, $v = x$. Then
$I = \displaystyle\int (\arcsin x)^2 dx = x(\arcsin x)^2 - 2\int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx$. To simplify the last integral, let $t = \arcsin x$ $[x = \sin t]$, so
$dt = \dfrac{1}{\sqrt{1-x^2}}\,dx$, and $\displaystyle\int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx = \int t\sin t\,dt$. To evaluate just the last integral, now let $U = t$, $dV = \sin t\,dt \;\Rightarrow$
$dU = dt$, $V = -\cos t$. Thus,

$$\begin{aligned}\int t\sin t\,dt &= -t\cos t + \int \cos t\,dt = -t\cos t + \sin t + C\\ &= -\arcsin x \cdot \frac{\sqrt{1-x^2}}{1} + x + C_1 \quad \text{[refer to the figure]}\end{aligned}$$



Returning to I, we get $I = x(\arcsin x)^2 + 2\sqrt{1-x^2}\,\arcsin x - 2x + C$,
where $C = -2C_1$.

27. First let $u = 1 + x^2$, $dv = e^{3x}\,dx \;\Rightarrow\; du = 2x\,dx$, $v = \frac{1}{3}e^{3x}$. Then
$I = \int(1+x^2)e^{3x}\,dx = \frac{1}{3}e^{3x}(1+x^2) - \frac{2}{3}\int xe^{3x}\,dx$. Next, let $U = x$, $dV = e^{3x}\,dx \;\Rightarrow\; dU = dx$, $V = \frac{1}{3}e^{3x}$, so
$\int xe^{3x}\,dx = \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\,dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C_1$. Substituting in the previous formula gives

$$\begin{aligned}I &= \tfrac{1}{3}e^{3x}(1+x^2) - \tfrac{2}{3}\left(\tfrac{1}{3}xe^{3x} - \tfrac{1}{9}e^{3x} + C_1\right) = \tfrac{1}{3}e^{3x} + \tfrac{1}{3}x^2e^{3x} - \tfrac{2}{9}xe^{3x} + \tfrac{2}{27}e^{3x} - \tfrac{2}{3}C_1\\ &= \tfrac{11}{27}e^{3x} - \tfrac{2}{9}xe^{3x} + \tfrac{1}{3}x^2e^{3x} + C, \text{ where } C = -\tfrac{2}{3}C_1\end{aligned}$$

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28. Let $u = \theta$, $dv = \sin 3\pi\theta\, d\theta \;\Rightarrow\; du = d\theta$, $v = -\dfrac{1}{3\pi}\cos 3\pi\theta$. By (6),

$$\int_0^{1/2} \theta \sin 3\pi\theta\, d\theta = \left[-\frac{1}{3\pi}\theta\cos 3\pi\theta\right]_0^{1/2} + \frac{1}{3\pi}\int_0^{1/2}\cos 3\pi\theta\, d\theta = (0+0) + \frac{1}{9\pi^2}\left[\sin 3\pi\theta\right]_0^{1/2}$$
$$= \frac{1}{9\pi^2}(-1-0) = -\frac{1}{9\pi^2}$$

29. Let $u = x$, $dv = 3^x\, dx \;\Rightarrow\; du = dx$, $v = \dfrac{1}{\ln 3}3^x$. By (6),

$$\int_0^1 x3^x\, dx = \left[\frac{1}{\ln 3}\, x\, 3^x\right]_0^1 - \frac{1}{\ln 3}\int_0^1 3^x\, dx = \left(\frac{3}{\ln 3} - 0\right) - \frac{1}{\ln 3}\left[\frac{1}{\ln 3}\, 3^x\right]_0^1 = \frac{3}{\ln 3} - \frac{1}{(\ln 3)^2}(3-1)$$
$$= \frac{3}{\ln 3} - \frac{2}{(\ln 3)^2}$$

30. Let $u = xe^x$, $dv = \dfrac{1}{(1+x)^2}\, dx \;\Rightarrow\; du = (xe^x + e^x)\, dx = e^x(x+1)\, dx$, $v = -\dfrac{1}{1+x}$. By (6),

$$\int_0^1 \frac{xe^x}{(1+x)^2}\, dx = \left[-\frac{xe^x}{1+x}\right]_0^1 - \int_0^1\left(-\frac{1}{1+x}\right)e^x(1+x)\, dx = \left(-\frac{e}{2} + 0\right) + \int_0^1 e^x\, dx = -\frac{1}{2}e + \left[e^x\right]_0^1$$
$$= -\tfrac{1}{2}e + e - 1 = \tfrac{1}{2}e - 1$$

31. Let $u = y$, $dv = \sinh y\, dy \;\Rightarrow\; du = dy$, $v = \cosh y$. By (6),

$$\int_0^2 y\sinh y\, dy = \left[y\cosh y\right]_0^2 - \int_0^2 \cosh y\, dy = 2\cosh 2 - 0 - \left[\sinh y\right]_0^2 = 2\cosh 2 - \sinh 2.$$

32. Let $u = \ln w$, $dv = w^2\, dw \;\Rightarrow\; du = \dfrac{1}{w}\, dw$, $v = \tfrac{1}{3}w^3$. By (6),

$\int_1^2 w^2 \ln w\, dw = \left[\tfrac{1}{3}w^3\ln w\right]_1^2 - \int_1^2 \tfrac{1}{3}w^2\, dw = \tfrac{8}{3}\ln 2 - 0 - \left[\tfrac{1}{9}w^3\right]_1^2 = \tfrac{8}{3}\ln 2 - \left(\tfrac{8}{9} - \tfrac{1}{9}\right) = \tfrac{8}{3}\ln 2 - \tfrac{7}{9}.$

33. Let $u = \ln R$, $dv = \dfrac{1}{R^2}\, dR \;\Rightarrow\; du = \dfrac{1}{R}\, dR$, $v = -\dfrac{1}{R}$. By (6),

$$\int_1^5 \frac{\ln R}{R^2}\, dR = \left[-\frac{1}{R}\ln R\right]_1^5 - \int_1^5 -\frac{1}{R^2}\, dR = -\tfrac{1}{5}\ln 5 - 0 - \left[\frac{1}{R}\right]_1^5 = -\tfrac{1}{5}\ln 5 - \left(\tfrac{1}{5} - 1\right) = \tfrac{4}{5} - \tfrac{1}{5}\ln 5.$$

34. First let $u = t^2$, $dv = \sin 2t\, dt \;\Rightarrow\; du = 2t\, dt$, $v = -\tfrac{1}{2}\cos 2t$. By (6),

$\int_0^{2\pi} t^2\sin 2t\, dt = \left[-\tfrac{1}{2}t^2\cos 2t\right]_0^{2\pi} + \int_0^{2\pi} t\cos 2t\, dt = -2\pi^2 + \int_0^{2\pi} t\cos 2t\, dt.$

Next let $U = t$, $dV = \cos 2t\, dt \;\Rightarrow\; dU = dt$, $V = \tfrac{1}{2}\sin 2t$. By (6) again,

$\int_0^{2\pi} t\cos 2t\, dt = \left[\tfrac{1}{2}t\sin 2t\right]_0^{2\pi} - \int_0^{2\pi}\tfrac{1}{2}\sin 2t\, dt = 0 - \left[-\tfrac{1}{4}\cos 2t\right]_0^{2\pi} = \tfrac{1}{4} - \tfrac{1}{4} = 0.$ Thus, $\int_0^{2\pi} t^2\sin 2t\, dt = -2\pi^2.$

35. $\sin 2x = 2\sin x\, \cos x$, so $\int_0^\pi x\,\sin x\,\cos x\, dx = \tfrac{1}{2}\int_0^\pi x\,\sin 2x\, dx$. Let $u = x$, $dv = \sin 2x\, dx \;\Rightarrow$

$du = dx$, $v = -\tfrac{1}{2}\cos 2x$. By (6),

$$\tfrac{1}{2}\int_0^\pi x\,\sin 2x\, dx = \tfrac{1}{2}\left[-\tfrac{1}{2}x\cos 2x\right]_0^\pi - \tfrac{1}{2}\int_0^\pi -\tfrac{1}{2}\cos 2x\, dx = -\tfrac{1}{4}\pi - 0 + \tfrac{1}{4}\left[\tfrac{1}{2}\sin 2x\right]_0^\pi = -\frac{\pi}{4}$$

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42. Let $u = \sin(t-s)$, $dv = e^s\,ds \Rightarrow du = -\cos(t-s)\,ds$, $v = e^s$. Then

$I = \int_0^t e^s \sin(t-s)\,ds = \left[e^s\sin(t-s)\right]_0^t + \int_0^t e^s\cos(t-s)\,ds = e^t\sin 0 - e^0\sin t + I_1$. For I_1, let $U = \cos(t-s)$,

$dV = e^s\,ds \Rightarrow dU = \sin(t-s)\,ds$, $V = e^s$. So $I_1 = \left[e^s\cos(t-s)\right]_0^t - \int_0^t e^s\sin(t-s)\,ds = e^t\cos 0 - e^0\cos t - I$.

Thus, $I = -\sin t + e^t - \cos t - I \Rightarrow 2I = e^t - \cos t - \sin t \Rightarrow I = \frac{1}{2}(e^t - \cos t - \sin t)$.

43. Let $t = \sqrt{x}$, so that $t^2 = x$ and $2t\,dt = dx$. Thus, $\int e^{\sqrt{x}}\,dx = \int e^t(2t)\,dt$. Now use parts with $u = t$, $dv = e^t\,dt$, $du = dt$, and $v = e^t$ to get $2\int te^t\,dt = 2te^t - 2\int e^t\,dt = 2te^t - 2e^t + C = 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$.

44. Let $t = \ln x$, so that $e^t = x$ and $e^t\,dt = dx$. Thus, $\int \cos(\ln x)\,dx = \int \cos t \cdot e^t\,dt = I$. Now use parts with $u = \cos t$, $dv = e^t\,dt$, $du = -\sin t\,dt$, and $v = e^t$ to get $\int e^t\cos t\,dt = e^t\cos t - \int -e^t\sin t\,dt = e^t\cos t + \int e^t\sin t\,dt$. Now use parts with $U = \sin t$, $dV = e^t\,dt$, $dU = \cos t\,dt$, and $V = e^t$ to get

$\int e^t\sin t\,dt = e^t\sin t - \int e^t\cos t\,dt$. Thus, $I = e^t\cos t + e^t\sin t - I \Rightarrow 2I = e^t\cos t + e^t\sin t \Rightarrow$

$I = \frac{1}{2}e^t\cos t + \frac{1}{2}e^t\sin t + C = \frac{1}{2}x\cos(\ln x) + \frac{1}{2}x\sin(\ln x) + C$.

45. Let $x = \theta^2$, so that $dx = 2\theta\,d\theta$. Thus, $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3\cos(\theta^2)\,d\theta = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^2\cos(\theta^2)\cdot\tfrac{1}{2}(2\theta\,d\theta) = \tfrac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx$. Now use parts with $u = x$, $dv = \cos x\,dx$, $du = dx$, $v = \sin x$ to get

$$\begin{aligned}\tfrac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx &= \tfrac{1}{2}\left(\left[x\sin x\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\sin x\,dx\right) = \tfrac{1}{2}\left[x\sin x + \cos x\right]_{\pi/2}^{\pi} \\ &= \tfrac{1}{2}(\pi\sin\pi + \cos\pi) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\sin\tfrac{\pi}{2} + \cos\tfrac{\pi}{2}\right) = \tfrac{1}{2}(\pi\cdot 0 - 1) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\cdot 1 + 0\right) = -\tfrac{1}{2} - \tfrac{\pi}{4}\end{aligned}$$

46. Let $x = \cos t$, so that $dx = -\sin t\,dt$. Thus,

$\int_0^\pi e^{\cos t}\sin 2t\,dt = \int_0^\pi e^{\cos t}(2\sin t\cos t)\,dt = \int_1^{-1} e^x\cdot 2x\,(-dx) = 2\int_{-1}^1 xe^x\,dx$. Now use parts with $u = x$, $dv = e^x\,dx$, $du = dx$, $v = e^x$ to get

$2\int_{-1}^1 xe^x\,dx = 2\left(\left[xe^x\right]_{-1}^1 - \int_{-1}^1 e^x\,dx\right) = 2\left(e^1 + e^{-1} - \left[e^x\right]_{-1}^1\right) = 2(e + e^{-1} - [e^1 - e^{-1}]) = 2(2e^{-1}) = 4/e$.

47. Let $y = 1 + x$, so that $dy = dx$. Thus, $\int x\ln(1+x)\,dx = \int (y-1)\ln y\,dy$. Now use parts with $u = \ln y$, $dv = (y-1)\,dy$, $du = \frac{1}{y}\,dy$, $v = \frac{1}{2}y^2 - y$ to get

$$\begin{aligned}\int (y-1)\ln y\,dy &= \left(\tfrac{1}{2}y^2 - y\right)\ln y - \int\left(\tfrac{1}{2}y - 1\right)dy = \tfrac{1}{2}y(y-2)\ln y - \tfrac{1}{4}y^2 + y + C \\ &= \tfrac{1}{2}(1+x)(x-1)\ln(1+x) - \tfrac{1}{4}(1+x)^2 + 1 + x + C,\end{aligned}$$

which can be written as $\frac{1}{2}(x^2-1)\ln(1+x) - \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{4} + C$.

48. Let $y = \ln x$, so that $dy = \dfrac{1}{x}\,dx$. Thus, $\displaystyle\int \frac{\arcsin(\ln x)}{x}\,dx = \int \arcsin y\,dy$. Now use parts with $u = \arcsin y$, $dv = dy$, $du = \dfrac{1}{\sqrt{1-y^2}}\,dy$, and $v = y$ to get

$$\int \arcsin y\,dy = y\arcsin y - \int \frac{y}{\sqrt{1-y^2}}\,dy = y\arcsin y + \sqrt{1-y^2} + C = (\ln x)\arcsin(\ln x) + \sqrt{1-(\ln x)^2} + C.$$