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(d) We substitute the results from Exercises 55 and 56 into the result from part (c):

$$1 = \lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = \lim_{n\to\infty} \frac{\dfrac{2\cdot 4\cdot 6\cdot\dots\cdot(2n)}{3\cdot 5\cdot 7\cdot\dots\cdot(2n+1)}}{\dfrac{1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\dots\cdot(2n)}\dfrac{\pi}{2}} = \lim_{n\to\infty}\left[\frac{2\cdot 4\cdot 6\cdot\dots\cdot(2n)}{3\cdot 5\cdot 7\cdot\dots\cdot(2n+1)}\right]\left[\frac{2\cdot 4\cdot 6\cdot\dots\cdot(2n)}{1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)}\left(\frac{2}{\pi}\right)\right]$$

$$= \lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\dots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\frac{2}{\pi} \qquad \text{[rearrange terms]}$$

Multiplying both sides by $\frac{\pi}{2}$ gives us the *Wallis product*:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\dots$$

(e) The area of the kth rectangle is k. At the $2n$th step, the area is increased from $2n-1$ to $2n$ by multiplying the width by $\dfrac{2n}{2n-1}$, and at the $(2n+1)$th step, the area is increased from $2n$ to $2n+1$ by multiplying the height by $\dfrac{2n+1}{2n}$. These two steps multiply the ratio of width to height by $\dfrac{2n}{2n-1}$ and $\dfrac{1}{(2n+1)/(2n)} = \dfrac{2n}{2n+1}$ respectively. So, by part (d), the limiting ratio is $\dfrac{2}{1}\cdot\dfrac{2}{3}\cdot\dfrac{4}{3}\cdot\dfrac{4}{5}\cdot\dfrac{6}{5}\cdot\dfrac{6}{7}\cdot\dots = \dfrac{\pi}{2}$.

81. Using the formula for volumes of rotation and the figure, we see that

Volume $= \int_0^d \pi b^2\,dy - \int_0^c \pi a^2\,dy - \int_c^d \pi[g(y)]^2\,dy = \pi b^2 d - \pi a^2 c - \int_c^d \pi[g(y)]^2\,dy$. Let $y = f(x)$,

which gives $dy = f'(x)\,dx$ and $g(y) = x$, so that $V = \pi b^2 d - \pi a^2 c - \pi\int_a^b x^2 f'(x)\,dx$.

Now integrate by parts with $u = x^2$, and $dv = f'(x)\,dx \;\Rightarrow\; du = 2x\,dx$, $v = f(x)$, and

$\int_a^b x^2 f'(x)\,dx = \left[x^2 f(x)\right]_a^b - \int_a^b 2x\,f(x)\,dx = b^2 f(b) - a^2 f(a) - \int_a^b 2x\,f(x)\,dx$, but $f(a) = c$ and $f(b) = d \;\Rightarrow$

$V = \pi b^2 d - \pi a^2 c - \pi\left[b^2 d - a^2 c - \int_a^b 2x\,f(x)\,dx\right] = \int_a^b 2\pi x\,f(x)\,dx$.

7.2 Trigonometric Integrals

The symbols $\overset{s}{=}$ and $\overset{c}{=}$ indicate the use of the substitutions $\{u = \sin x, du = \cos x\,dx\}$ and $\{u = \cos x, du = -\sin x\,dx\}$, respectively.

1. $\int \sin^3 x\cos^2 x\,dx = \int \sin^2 x\cos^2 x\sin x\,dx = \int (1-\cos^2 x)\cos^2 x\sin x\,dx \overset{c}{=} \int (1-u^2)u^2\,(-du)$

$\quad = \int (u^4 - u^2)\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$

2. $\int \cos^6 y\sin^3 y\,dy = \int \cos^6 y\sin^2 y\sin y\,dy = \int \cos^6 y(1-\cos^2 y)\sin y\,dy \overset{c}{=} \int u^6(1-u^2)\,(-du)$

$\quad = \int (u^8 - u^6)\,du = \frac{1}{9}u^9 - \frac{1}{7}u^7 + C = \frac{1}{9}\cos^9 y - \frac{1}{7}\cos^7 y + C$

3. $\displaystyle\int_0^{\pi/2} \cos^9 x\sin^5 x\,dx = \int_0^{\pi/2} \cos^9 x\,(\sin^2 x)^2\sin x\,dx = \int_0^{\pi/2}\cos^9 x\,(1-\cos^2 x)^2\sin x\,dx$

$$\overset{c}{=} \int_1^0 u^9(1-u^2)^2\,(-du) = \int_0^1 u^9(1-2u^2+u^4)\,du = \int_0^1 (u^9 - 2u^{11} + u^{13})\,du$$

$$= \left[\tfrac{1}{10}u^{10} - \tfrac{1}{6}u^{12} + \tfrac{1}{14}u^{14}\right]_0^1 = \left(\tfrac{1}{10} - \tfrac{1}{6} + \tfrac{1}{14}\right) - 0 = \tfrac{1}{210}$$

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4. $\int_0^{\pi/4} \sin^5 x\, dx = \int_0^{\pi/4} (\sin^2 x)^2 \sin x\, dx = \int_0^{\pi/4} (1-\cos^2 x)^2 \sin x\, dx \overset{c}{=} \int_1^{1/\sqrt{2}} (1-u^2)^2\,(-du)$

$= \int_{1/\sqrt{2}}^{1} (1-2u^2+u^4)\, du = \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_{1/\sqrt{2}}^{1} = \left(1 - \frac{2}{3} + \frac{1}{5}\right) - \left(\frac{1}{\sqrt{2}} - \frac{2}{3\sqrt{8}} + \frac{1}{5\sqrt{32}}\right)$

$= \frac{8}{15} - \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{6} + \frac{\sqrt{2}}{40}\right) = \frac{8}{15} - \frac{43\sqrt{2}}{120}$

5. $\int \sin^5(2t)\cos^2(2t)\, dt = \int \sin^4(2t)\, \cos^2(2t)\, \sin(2t)\, dt = \int [1-\cos^2(2t)]^2 \cos^2(2t)\, \sin(2t)\, dt$

$= \int (1-u^2)^2\, u^2\, \left(-\frac{1}{2}\, du\right) \qquad [u = \cos(2t),\, du = -2\sin(2t)\, dt]$

$= -\frac{1}{2}\int (u^4 - 2u^2 + 1)u^2\, du = -\frac{1}{2}\int (u^6 - 2u^4 + u^2)\, du$

$= -\frac{1}{2}\left(\frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3\right) + C = -\frac{1}{14}\cos^7(2t) + \frac{1}{5}\cos^5(2t) - \frac{1}{6}\cos^3(2t) + C$

6. $\int \cos^3\left(\frac{t}{2}\right)\sin^2\left(\frac{t}{2}\right) dt = \int \cos^2\left(\frac{t}{2}\right)\sin^2\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right) dt = \int \left(1-\sin^2\left(\frac{t}{2}\right)\right)\sin^2\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right) dt$

$= \int (1-u^2)u^2(2\, du) \qquad \left[u = \sin\left(\frac{t}{2}\right),\, du = \frac{1}{2}\cos\left(\frac{t}{2}\right) dt\right]$

$= 2\int (u^2 - u^4)\, du = 2\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C = \frac{2}{3}\sin^3\left(\frac{t}{2}\right) - \frac{2}{5}\sin^5\left(\frac{t}{2}\right) + C$

7. $\int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\, d\theta$ [half-angle identity]

$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2}+0\right) - (0+0)\right] = \frac{\pi}{4}$

8. $\int_0^{\pi/4} \sin^2(2\theta)\, d\theta = \int_0^{\pi/4} \frac{1}{2}(1-\cos 4\theta)\, d\theta$ [half-angle identity]

$= \frac{1}{2}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{1}{2}\left[\left(\frac{\pi}{4} - 0\right) - 0\right] = \frac{\pi}{8}$

9. $\int_0^{\pi} \cos^4(2t)\, dt = \int_0^{\pi} [\cos^2(2t)]^2\, dt = \int_0^{\pi} \left[\frac{1}{2}(1+\cos(2\cdot 2t))\right]^2 dt$ [half-angle identity]

$= \frac{1}{4}\int_0^{\pi} [1 + 2\cos 4t + \cos^2(4t)]\, dt = \frac{1}{4}\int_0^{\pi} [1 + 2\cos 4t + \frac{1}{2}(1+\cos 8t)]\, dt$

$= \frac{1}{4}\int_0^{\pi} \left(\frac{3}{2} + 2\cos 4t + \frac{1}{2}\cos 8t\right) dt = \frac{1}{4}\left[\frac{3}{2}t + \frac{1}{2}\sin 4t + \frac{1}{16}\sin 8t\right]_0^{\pi} = \frac{1}{4}\left[\left(\frac{3}{2}\pi + 0 + 0\right) - 0\right] = \frac{3}{8}\pi$

10. $\int_0^{\pi} \sin^2 t\, \cos^4 t\, dt = \frac{1}{4}\int_0^{\pi} (4\sin^2 t\, \cos^2 t)\cos^2 t\, dt = \frac{1}{4}\int_0^{\pi} (2\sin t\, \cos t)^2\, \frac{1}{2}(1+\cos 2t)\, dt$

$= \frac{1}{8}\int_0^{\pi} (\sin 2t)^2 (1+\cos 2t)\, dt = \frac{1}{8}\int_0^{\pi} (\sin^2 2t + \sin^2 2t\, \cos 2t)\, dt$

$= \frac{1}{8}\int_0^{\pi} \sin^2 2t\, dt + \frac{1}{8}\int_0^{\pi} \sin^2 2t\, \cos 2t\, dt = \frac{1}{8}\int_0^{\pi} \frac{1}{2}(1-\cos 4t)\, dt + \frac{1}{8}\left[\frac{1}{3}\cdot\frac{1}{2}\sin^3 2t\right]_0^{\pi}$

$= \frac{1}{16}\left[t - \frac{1}{4}\sin 4t\right]_0^{\pi} + \frac{1}{8}(0-0) = \frac{1}{16}\left[(\pi - 0) - 0\right] = \frac{\pi}{16}$

11. $\int_0^{\pi/2} \sin^2 x\, \cos^2 x\, dx = \int_0^{\pi/2} \frac{1}{4}(4\sin^2 x\, \cos^2 x)\, dx = \int_0^{\pi/2} \frac{1}{4}(2\sin x\, \cos x)^2 dx = \frac{1}{4}\int_0^{\pi/2} \sin^2 2x\, dx$

$= \frac{1}{4}\int_0^{\pi/2} \frac{1}{2}(1-\cos 4x)\, dx = \frac{1}{8}\int_0^{\pi/2}(1-\cos 4x)\, dx = \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/2} = \frac{1}{8}\left(\frac{\pi}{2}\right) = \frac{\pi}{16}$

12. $\int_0^{\pi/2} (2-\sin\theta)^2\, d\theta = \int_0^{\pi/2} (4 - 4\sin\theta + \sin^2\theta)\, d\theta = \int_0^{\pi/2} \left[4 - 4\sin\theta + \frac{1}{2}(1-\cos 2\theta)\right] d\theta$

$= \int_0^{\pi/2} \left(\frac{9}{2} - 4\sin\theta - \frac{1}{2}\cos 2\theta\right) d\theta = \left[\frac{9}{2}\theta + 4\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2}$

$= \left(\frac{9\pi}{4} + 0 - 0\right) - (0 + 4 - 0) = \frac{9}{4}\pi - 4$

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13. $\int \sqrt{\cos\theta}\sin^3\theta\,d\theta = \int \sqrt{\cos\theta}\sin^2\theta\,\sin\theta\,d\theta = \int(\cos\theta)^{1/2}(1-\cos^2\theta)\sin\theta\,d\theta$
$\overset{c}{=} \int u^{1/2}(1-u^2)\,(-du) = \int(u^{5/2}-u^{1/2})\,du$
$= \frac{2}{7}u^{7/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{7}(\cos\theta)^{7/2} - \frac{2}{3}(\cos\theta)^{3/2} + C$

14. $\displaystyle\int\left(1+\sqrt[3]{\sin t}\right)\cos^3 t\,dt = \int\left(1+(\sin t)^{1/3}\right)\cos^2 t\cos t\,dt = \int\left(1+(\sin t)^{1/3}\right)(1-\sin^2 t)\cos t\,dt$
$\overset{s}{=} \int(1+u^{1/3})(1-u^2)\,du = \int(1-u^2+u^{1/3}-u^{7/3})\,du$
$= u - \frac{1}{3}u^3 + \frac{3}{4}u^{4/3} - \frac{3}{10}u^{10/3} + C$
$= \sin t - \frac{1}{3}\sin^3 t + \frac{3}{4}\sqrt[3]{\sin^4 t} - \frac{3}{10}\sqrt[3]{\sin^{10} t} + C$

15. $\displaystyle\int \sin x\,\sec^5 x\,dx = \int\frac{\sin x}{\cos^5 x}\,dx \overset{c}{=} \int\frac{1}{u^5}\,(-du) = \frac{1}{4u^4} + C = \frac{1}{4\cos^4 x} + C = \frac{1}{4}\sec^4 x + C$

16. $\displaystyle\int \csc^5\theta\,\cos^3\theta\,d\theta = \int\frac{\cos^2\theta}{\sin^5\theta}\cos\theta\,d\theta = \int\frac{1-\sin^2\theta}{\sin^5\theta}\cos\theta\,d\theta \overset{s}{=} \int\frac{1-u^2}{u^5}\,du = \int(u^{-5}-u^{-3})\,du$
$\displaystyle= -\frac{1}{4}u^{-4} + \frac{1}{2}u^{-2} + C = -\frac{1}{4\sin^4\theta} + \frac{1}{2\sin^2\theta} + C$

Alternate solution:

$\displaystyle\int \csc^5\theta\cos^3\theta\,d\theta = \int\left(\frac{\cos^3\theta}{\sin^3\theta}\right)\left(\frac{1}{\sin^2\theta}\right)d\theta = \int\cot^3\theta\,\csc^2\theta\,d\theta$
$= \int u^3\,(-du) \qquad [u=\cot\theta,\ du=-\csc^2\theta\,d\theta]$
$= -\int u^3\,du = -\frac{1}{4}u^4 + C = -\frac{1}{4}\cot^4\theta + C$

17. $\displaystyle\int \cot x\,\cos^2 x\,dx = \int\frac{\cos x}{\sin x}(1-\sin^2 x)\,dx$
$\displaystyle\overset{s}{=} \int\frac{1-u^2}{u}\,du = \int\left(\frac{1}{u}-u\right)du = \ln|u| - \frac{1}{2}u^2 + C = \ln|\sin x| - \frac{1}{2}\sin^2 x + C$

18. $\displaystyle\int \tan^2 x\,\cos^3 x\,dx = \int\frac{\sin^2 x}{\cos^2 x}\cos^3 x\,dx = \int\sin^2 x\,\cos x\,dx \overset{s}{=} \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C$

19. $\int \sin^2 x\,\sin 2x\,dx = \int\sin^2 x\,(2\sin x\,\cos x)\,dx \overset{s}{=} \int 2u^3\,du = \frac{1}{2}u^4 + C = \frac{1}{2}\sin^4 x + C$

20. $\int \sin x\,\cos\left(\frac{1}{2}x\right)dx = \int\sin\left(2\cdot\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)dx = \int 2\sin\left(\frac{1}{2}x\right)\cos^2\left(\frac{1}{2}x\right)dx$
$= \int 2u^2\,(-2\,du) \qquad [u=\cos\left(\frac{1}{2}x\right),\ du=-\frac{1}{2}\sin\left(\frac{1}{2}x\right)dx]$
$= -\frac{4}{3}u^3 + C = -\frac{4}{3}\cos^3\left(\frac{1}{2}x\right) + C$

21. $\int \tan x\,\sec^3 x\,dx = \int\tan x\,\sec x\,\sec^2 x\,dx = \int u^2\,du \qquad [u=\sec x,\ du=\sec x\,\tan x\,dx]$
$= \frac{1}{3}u^3 + C = \frac{1}{3}\sec^3 x + C$

22. $\int \tan^2\theta\,\sec^4\theta\,d\theta = \int\tan^2\theta\,\sec^2\theta\,\sec^2\theta\,d\theta = \int\tan^2\theta\,(\tan^2\theta+1)\,\sec^2\theta\,d\theta$
$= \int u^2(u^2+1)\,du \qquad [u=\tan\theta,\ du=\sec^2\theta\,d\theta]$
$= \int(u^4+u^2)\,du = \frac{1}{5}u^5 + \frac{1}{3}u^3 + C = \frac{1}{5}\tan^5\theta + \frac{1}{3}\tan^3\theta + C$

23. $\int \tan^2 x\,dx = \int(\sec^2 x - 1)\,dx = \tan x - x + C$

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24. $\int(\tan^2 x + \tan^4 x)\,dx = \int \tan^2 x\,(1+\tan^2 x)\,dx = \int \tan^2 x\,\sec^2 x\,dx = \int u^2\,du \qquad [u = \tan x,\ du = \sec^2 x\,dx]$

$= \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C$

25. Let $u = \tan x$. Then $du = \sec^2 x\,dx$, so

$$\begin{aligned}\int \tan^4 x \sec^6 x\,dx &= \int \tan^4 x \sec^4 x\,(\sec^2 x\,dx) = \int \tan^4 x(1+\tan^2 x)^2\,(\sec^2 x\,dx)\\ &= \int u^4(1+u^2)^2\,du = \int(u^8 + 2u^6 + u^4)\,du\\ &= \tfrac{1}{9}u^9 + \tfrac{2}{7}u^7 + \tfrac{1}{5}u^5 + C = \tfrac{1}{9}\tan^9 x + \tfrac{2}{7}\tan^7 x + \tfrac{1}{5}\tan^5 x + C\end{aligned}$$

26. $\int_0^{\pi/4}\sec^6\theta\,\tan^6\theta\,d\theta = \int_0^{\pi/4}\tan^6\theta\,\sec^4\theta\,\sec^2\theta\,d\theta = \int_0^{\pi/4}\tan^6\theta(1+\tan^2\theta)^2\sec^2\theta\,d\theta$

$= \int_0^1 u^6(1+u^2)^2\,du \qquad \begin{bmatrix} u = \tan\theta, \\ du = \sec^2\theta\,d\theta\end{bmatrix}$

$= \int_0^1 u^6(u^4 + 2u^2 + 1)\,du = \int_0^1(u^{10} + 2u^8 + u^6)\,du$

$= \left[\frac{1}{11}u^{11} + \frac{2}{9}u^9 + \frac{1}{7}u^7\right]_0^1 = \frac{1}{11} + \frac{2}{9} + \frac{1}{7} = \frac{63+154+99}{693} = \frac{316}{693}$

27. $\int \tan^3 x\,\sec x\,dx = \int \tan^2 x\,\sec x\,\tan x\,dx = \int(\sec^2 x - 1)\,\sec x\,\tan x\,dx$

$= \int(u^2-1)\,du \quad [u = \sec x, du = \sec x\,\tan x\,dx] \quad = \frac{1}{3}u^3 - u + C = \frac{1}{3}\sec^3 x - \sec x + C$

28. Let $u = \sec x$, so $du = \sec x \tan x\,dx$. Thus,

$$\begin{aligned}\int \tan^5 x \sec^3 x\,dx &= \int \tan^4 x \sec^2 x\,(\sec x \tan x)\,dx = \int(\sec^2 x - 1)^2\sec^2 x\,(\sec x\tan x\,dx)\\ &= \int(u^2-1)^2 u^2\,du = \int(u^6 - 2u^4 + u^2)\,du\\ &= \tfrac{1}{7}u^7 - \tfrac{2}{5}u^5 + \tfrac{1}{3}u^3 + C = \tfrac{1}{7}\sec^7 x - \tfrac{2}{5}\sec^5 x + \tfrac{1}{3}\sec^3 x + C\end{aligned}$$

29. $\int \tan^3 x\,\sec^6 x\,dx = \int \tan^3 x\,\sec^4 x\,\sec^2 x\,dx = \int \tan^3 x\,(1+\tan^2 x)^2\sec^2 x\,dx$

$= \int u^3(1+u^2)^2\,du \qquad \begin{bmatrix} u = \tan x, \\ du = \sec^2 x\,dx\end{bmatrix}$

$= \int u^3(u^4 + 2u^2 + 1)\,du = \int(u^7 + 2u^5 + u^3)\,du$

$= \frac{1}{8}u^8 + \frac{1}{3}u^6 + \frac{1}{4}u^4 + C = \frac{1}{8}\tan^8 x + \frac{1}{3}\tan^6 x + \frac{1}{4}\tan^4 x + C$

30. $\int_0^{\pi/4}\tan^4 t\,dt = \int_0^{\pi/4}\tan^2 t\,(\sec^2 t - 1)\,dt = \int_0^{\pi/4}\tan^2 t\,\sec^2 t\,dt - \int_0^{\pi/4}\tan^2 t\,dt$

$= \int_0^1 u^2\,du\ [u = \tan t]\ - \int_0^{\pi/4}(\sec^2 t - 1)\,dt = \left[\frac{1}{3}u^3\right]_0^1 - \Big[\tan t - t\Big]_0^{\pi/4}$

$= \frac{1}{3} - \left[\left(1 - \frac{\pi}{4}\right) - 0\right] = \frac{\pi}{4} - \frac{2}{3}$

31. $\int \tan^5 x\,dx = \int(\sec^2 x - 1)^2\,\tan x\,dx = \int \sec^4 x\,\tan x\,dx - 2\int \sec^2 x\,\tan x\,dx + \int \tan x\,dx$

$= \int \sec^3 x\,\sec x\,\tan x\,dx - 2\int \tan x\,\sec^2 x\,dx + \int \tan x\,dx$

$= \frac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C \quad [\text{or } \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C]$

32. $\int \tan^2 x\,\sec x\,dx = \int(\sec^2 x - 1)\,\sec x\,dx = \int \sec^3 x\,dx - \int \sec x\,dx$

$= \frac{1}{2}(\sec x\,\tan x + \ln|\sec x + \tan x|) - \ln|\sec x + \tan x| + C \qquad$ [by Example 8 and (1)]

$= \frac{1}{2}(\sec x\,\tan x - \ln|\sec x + \tan x|) + C$