SOLUTION TO HW #10, PART IV
ANOTHER MIXING PROBLEM

Let $y = y(t)$ = the amount (lbs) of salt in the tank at time t minutes.

Let $V(t)$ = the volume (gal) of water in the tank at time t minutes.

Then, $V(0) = 100$ gallons and $y(0) = 0$ lbs. of salt.

Also, $V(t) = 100 + 5t - 3t = 100 + 2t$ gallons at time t.

Let r_{IN} = the rate (lbs/min) at which salt is entering the tank at time t.

Let r_{OUT} = the rate (lbs/min) at which salt is leaving the tank at time t.

dy/dt = the rate (lbs/min) at which the amount of salt in the tank is changing at time t.

Thus, $dy/dt = r_{IN} - r_{OUT}$

$r_{IN} = (0.5 \text{ lbs/gal}) \times (5 \text{ gal/min}) = 2.5 \text{ lbs/min}$.

$r_{OUT} = \left(\frac{y(t)}{V(t)} \text{ lbs/gal} \right) \left(3 \text{ gal/min} \right) = \left(\frac{3y}{100+2t} \text{ lbs/min} \right)$

So, $\frac{dy}{dt} = \left(2.5 - \frac{3y}{100+2t}\right)$ lbs/min.

[Adding $\frac{3y}{100+2t}$]

(*) $\frac{dy}{dt} + \left(\frac{3}{100+2t}\right)y = 2.5 = \frac{5}{2}$

This is a Linear First-ORDER DIFFERENTIAL EQUATION and we seek the solution y such that $y(0)=0$.

The Integrating factor is $e^{\int \frac{3}{100+2t}dt}$.

$$\int \frac{3}{100+2t}dt = \frac{3}{2}\int \frac{1}{u}du = \frac{3}{2}\ln u$$

$$= \ln u^{3/2} = \ln\left((100+2t)^{3/2}\right).$$

Let $u = 100+2t$
$du = 2dt$
$dt = \frac{1}{2}du$

So, the Integrating Factor is $e^{\ln((100+2t)^{3/2})} = \underline{\underline{(100+2t)^{3/2}}}$.

[We multiply both sides of the differential equation at (*) above by the integrating factor.]

$$(100+2t)^{3/2}\cdot\frac{dy}{dt} + 3(100+2t)^{\frac{1}{2}}y = \frac{5}{2}(100+2t)^{3/2}.$$

$$\left[(100+2t)^{3/2}y\right]' = \frac{5}{2}(100+2t)^{3/2}$$

$$(100+2t)^{3/2}y = \int\left[(100+2t)^{3/2}y\right]'dy = \frac{5}{2}\int(100+2t)^{3/2}dt$$

$$(100+2t)^{3/2}y = \frac{5}{2}\left(\frac{1}{2}\right)\int u^{3/2} + C$$

SOLUTION TO "ANOTHER MIXING PROBLEM" (p.3)

$$(100+2t)^{3/2}\,y = \frac{5}{4}\cdot\left(\frac{2}{5}u^{5/2}\right)+C$$

$$(100+2t)^{3/2}\,y = \frac{1}{2}(100+2t)^{5/2}+C$$

[Multiplying both sides by $(100+2t)^{-3/2}$]

$$y = \frac{1}{2}(100+2t)+C(100+2t)^{-3/2}.$$

[Solving for C]

When $t=0$, $y(0)=0$, so, setting $t=0$,

$$0 = \frac{1}{2}(100)+C(100)^{-3/2} = 50+\frac{C}{100^{3/2}}$$

$$0 = 50+\frac{C}{1000} \Rightarrow 0 = 50{,}000+C$$

$\therefore C = -50{,}000$ and so,

$$y = 50+t-\frac{50{,}000}{(100+2t)^{3/2}}.$$

The tank is full when $V(t)=200$.

$\Rightarrow 100+2t=200 \Rightarrow 2t=100 \Rightarrow t=50$

$\boxed{\text{(a) At time } t=50 \text{ minutes, the tank is full.}}$

(b) at time $t=50$, $y = 50+50-\dfrac{50{,}000}{(100+100)^{3/2}}$

$$y = 100-\frac{50{,}000}{(200)^{3/2}} = 82.3223$$

$\boxed{\text{(b) When the tank is full, the tank contains 82.32 lbs of salt.}}$