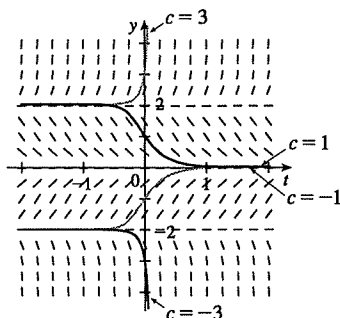


HW #10, Sec 9.2 Solutions

17.



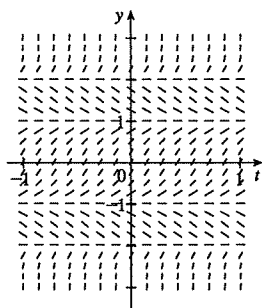
The direction field is for the differential equation $y' = y^3 - 4y$.

$L = \lim_{t \rightarrow \infty} y(t)$ exists for $-2 \leq c \leq 2$;

$L = \pm 2$ for $c = \pm 2$ and $L = 0$ for $-2 < c < 2$.

For other values of c , L does not exist.

18.



Note that when $f(y) = 0$ on the graph in the text, we have $y' = f(y) = 0$; so we get horizontal segments at $y = \pm 1, \pm 2$. We get segments with negative slopes only for $1 < |y| < 2$. All other segments have positive slope. For the limiting behavior of solutions:

- If $y(0) > 2$, then $\lim_{t \rightarrow \infty} y = \infty$ and $\lim_{t \rightarrow -\infty} y = 2$.
- If $1 < y(0) < 2$, then $\lim_{t \rightarrow \infty} y = 1$ and $\lim_{t \rightarrow -\infty} y = 2$.
- If $-1 < y(0) < 1$, then $\lim_{t \rightarrow \infty} y = 1$ and $\lim_{t \rightarrow -\infty} y = -1$.
- If $-2 < y(0) < -1$, then $\lim_{t \rightarrow \infty} y = -2$ and $\lim_{t \rightarrow -\infty} y = -1$.
- If $y < -2$, then $\lim_{t \rightarrow \infty} y = -2$ and $\lim_{t \rightarrow -\infty} y = -\infty$.

19. (a) $y' = F(x, y) = y$ and $y(0) = 1 \Rightarrow x_0 = 0, y_0 = 1$.

(i) $h = 0.4$ and $y_1 = y_0 + hF(x_0, y_0) \Rightarrow y_1 = 1 + 0.4 \cdot 1 = 1.4$. $x_1 = x_0 + h = 0 + 0.4 = 0.4$,

so $y_1 = y(0.4) = 1.4$.

(ii) $h = 0.2 \Rightarrow x_1 = 0.2$ and $x_2 = 0.4$, so we need to find y_2 .

$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.2y_0 = 1 + 0.2 \cdot 1 = 1.2$,

$y_2 = y_1 + hF(x_1, y_1) = 1.2 + 0.2y_1 = 1.2 + 0.2 \cdot 1.2 = 1.44$.

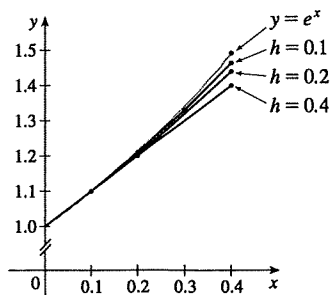
(iii) $h = 0.1 \Rightarrow x_4 = 0.4$, so we need to find y_4 . $y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1y_0 = 1 + 0.1 \cdot 1 = 1.1$,

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1y_1 = 1.1 + 0.1 \cdot 1.1 = 1.21$,

$y_3 = y_2 + hF(x_2, y_2) = 1.21 + 0.1y_2 = 1.21 + 0.1 \cdot 1.21 = 1.331$,

$y_4 = y_3 + hF(x_3, y_3) = 1.331 + 0.1y_3 = 1.331 + 0.1 \cdot 1.331 = 1.4641$.

(b)

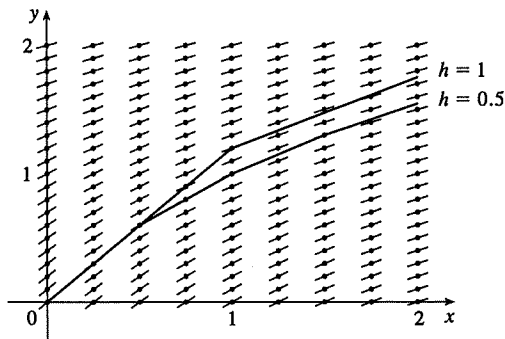


We see that the estimates are underestimates since they are all below the graph of $y = e^x$.

- (c) (i) For $h = 0.4$: (exact value) - (approximate value) = $e^{0.4} - 1.4 \approx 0.0918$
 (ii) For $h = 0.2$: (exact value) - (approximate value) = $e^{0.4} - 1.44 \approx 0.0518$
 (iii) For $h = 0.1$: (exact value) - (approximate value) = $e^{0.4} - 1.4641 \approx 0.0277$

Each time the step size is halved, the error estimate also appears to be halved (approximately).

20.



As x increases, the slopes decrease and all of the estimates are above the true values. Thus, all of the estimates are overestimates.

21. $h = 0.5$, $x_0 = 1$, $y_0 = 0$, and $F(x, y) = y - 2x$.

Note that $x_1 = x_0 + h = 1 + 0.5 = 1.5$, $x_2 = 2$, and $x_3 = 2.5$.

$$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.5F(1, 0) = 0.5[0 - 2(1)] = -1.$$

$$y_2 = y_1 + hF(x_1, y_1) = -1 + 0.5F(1.5, -1) = -1 + 0.5[-1 - 2(1.5)] = -3.$$

$$y_3 = y_2 + hF(x_2, y_2) = -3 + 0.5F(2, -3) = -3 + 0.5[-3 - 2(2)] = -6.5.$$

$$y_4 = y_3 + hF(x_3, y_3) = -6.5 + 0.5F(2.5, -6.5) = -6.5 + 0.5[-6.5 - 2(2.5)] = -12.25.$$

22. $h = 0.2$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = x^2y - \frac{1}{2}y^2$. Note that $x_1 = x_0 + h = 0 + 0.2 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, and $x_5 = 1$.

$$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.2F(0, 1) = 1 + 0.2 \left[0^2(1) - \frac{1}{2}(1)^2 \right] = 1 + 0.2 \left(-\frac{1}{2} \right) = 0.9.$$

$$y_2 = y_1 + hF(x_1, y_1) = 0.9 + 0.2F(0.2, 0.9) = 0.9 + 0.2 \left[(0.2)^2(0.9) - \frac{1}{2}(0.9)^2 \right] = 0.8262.$$

$$y_3 = y_2 + hF(x_2, y_2) = 0.8262 + 0.2F(0.4, 0.8262) = 0.8262 + 0.2 \left[(0.4)^2(0.8262) - \frac{1}{2}(0.8262)^2 \right] = 0.784377756.$$

$$y_4 = y_3 + hF(x_3, y_3) = 0.784377756 + 0.2F(0.6, 0.784377756) \approx 0.779328108.$$

$$y_5 = y_4 + hF(x_4, y_4) \approx 0.779328108 + 0.2F(0.8, 0.779328108) \approx 0.818346876.$$

Thus, $y(1) \approx 0.8183$.

23. $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = y + xy$.

Note that $x_1 = x_0 + h = 0 + 0.1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, and $x_4 = 0.4$.

$$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1F(0, 1) = 1 + 0.1[1 + (0)(1)] = 1.1.$$

$$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1F(0.1, 1.1) = 1.1 + 0.1[1.1 + (0.1)(1.1)] = 1.221.$$

$$y_3 = y_2 + hF(x_2, y_2) = 1.221 + 0.1F(0.2, 1.221) = 1.221 + 0.1[1.221 + (0.2)(1.221)] = 1.36752.$$

$$y_4 = y_3 + hF(x_3, y_3) = 1.36752 + 0.1F(0.3, 1.36752) = 1.36752 + 0.1[1.36752 + (0.3)(1.36752)] \\ = 1.5452976.$$

$$y_5 = y_4 + hF(x_4, y_4) = 1.5452976 + 0.1F(0.4, 1.5452976) \\ = 1.5452976 + 0.1[1.5452976 + (0.4)(1.5452976)] = 1.761639264.$$

Thus, $y(0.5) \approx 1.7616$.