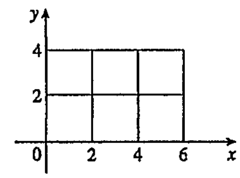
HW #12, SECTION 15.1 SOLUTIONS

15 ☐ MULTIPLE INTEGRALS

15.1 Double Integrals over Rectangles

1. (a) The subrectangles are shown in the figure.



The surface is the graph of $f(x,y) = xy$ and $\Delta A = 4$, so we estimate

$$V \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f(x_i, y_j)\,\Delta A$$

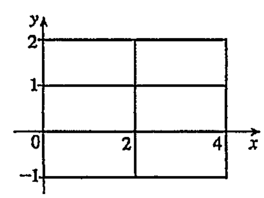
$$= f(2,2)\,\Delta A + f(2,4)\,\Delta A + f(4,2)\,\Delta A + f(4,4)\,\Delta A + f(6,2)\,\Delta A + f(6,4)\,\Delta A$$

$$= 4(4) + 8(4) + 8(4) + 16(4) + 12(4) + 24(4) = 288$$

(b) $V \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\,\Delta A = f(1,1)\,\Delta A + f(1,3)\,\Delta A + f(3,1)\,\Delta A + f(3,3)\,\Delta A + f(5,1)\,\Delta A + f(5,3)\,\Delta A$

$$= 1(4) + 3(4) + 3(4) + 9(4) + 5(4) + 15(4) = 144$$

2. (a) The subrectangles are shown in the figure.



Here $\Delta A = 2$ and we estimate

$$\iint_R (1 - xy^2)\,dA \approx \sum_{i=1}^{2} \sum_{j=1}^{3} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

$$= f(2,-1)\,\Delta A + f(2,0)\,\Delta A + f(2,1)\,\Delta A + f(4,-1)\,\Delta A + f(4,0)\,\Delta A + f(4,1)\,\Delta A$$

$$= (-1)(2) + 1(2) + (-1)(2) + (-3)(2) + 1(2) + (-3)(2) = -12$$

(b) $\iint_R (1 - xy^2)\,dA \approx \sum_{i=1}^{2} \sum_{j=1}^{3} f(x_{ij}^*, y_{ij}^*)\,\Delta A$

$$= f(0,0)\,\Delta A + f(0,1)\,\Delta A + f(0,2)\,\Delta A + f(2,0)\,\Delta A + f(2,1)\,\Delta A + f(2,2)\,\Delta A$$

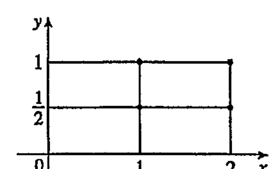
$$= 1(2) + 1(2) + 1(2) + 1(2) + (-1)(2) + (-7)(2) = -8$$

3. (a) The subrectangles are shown in the figure. Since $\Delta A = 1 \cdot \frac{1}{2} = \frac{1}{2}$, we estimate

$$\iint_R xe^{-xy}\,dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

$$= f\left(1, \tfrac{1}{2}\right)\,\Delta A + f(1,1)\,\Delta A + f\left(2, \tfrac{1}{2}\right)\,\Delta A + f(2,1)\,\Delta A$$

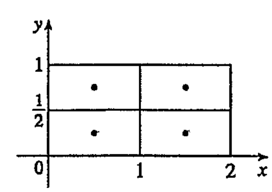
$$= e^{-1/2}\left(\tfrac{1}{2}\right) + e^{-1}\left(\tfrac{1}{2}\right) + 2e^{-1}\left(\tfrac{1}{2}\right) + 2e^{-2}\left(\tfrac{1}{2}\right) \approx 0.990$$



(b) $\iint_R xe^{-xy}\,dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\,\Delta A$

$$= f\left(\tfrac{1}{2}, \tfrac{1}{4}\right)\,\Delta A + f\left(\tfrac{1}{2}, \tfrac{3}{4}\right)\,\Delta A + f\left(\tfrac{3}{2}, \tfrac{1}{4}\right)\,\Delta A + f\left(\tfrac{3}{2}, \tfrac{3}{4}\right)\,\Delta A$$

$$= \tfrac{1}{2}e^{-1/8}\left(\tfrac{1}{2}\right) + \tfrac{1}{2}e^{-3/8}\left(\tfrac{1}{2}\right) + \tfrac{3}{2}e^{-3/8}\left(\tfrac{1}{2}\right) + \tfrac{3}{2}e^{-9/8}\left(\tfrac{1}{2}\right) \approx 1.151$$



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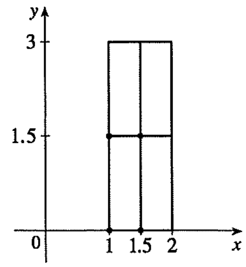
4. (a) The subrectangles are shown in the figure.

The surface is the graph of $f(x,y) = 1 + x^2 + 3y$ and $\Delta A = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$, so we estimate

$$V = \iint_R (1 + x^2 + 3y)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$

$$= f(1,0)\, \Delta A + f\left(1, \tfrac{3}{2}\right) \Delta A + f\left(\tfrac{3}{2}, 0\right) \Delta A + f\left(\tfrac{3}{2}, \tfrac{3}{2}\right) \Delta A$$

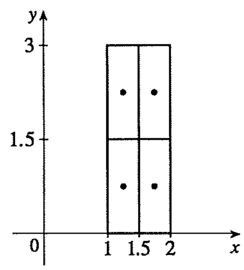
$$= 2\left(\tfrac{3}{4}\right) + \tfrac{13}{2}\left(\tfrac{3}{4}\right) + \tfrac{13}{4}\left(\tfrac{3}{4}\right) + \tfrac{31}{4}\left(\tfrac{3}{4}\right) = \tfrac{39}{2}\left(\tfrac{3}{4}\right) = \tfrac{117}{8} = 14.625$$



(b) $$V = \iint_R (1 + x^2 + 3y)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right) \Delta A$$

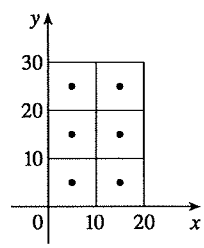
$$= f\left(\tfrac{5}{4}, \tfrac{3}{4}\right) \Delta A + f\left(\tfrac{5}{4}, \tfrac{9}{4}\right) \Delta A + f\left(\tfrac{7}{4}, \tfrac{3}{4}\right) \Delta A + f\left(\tfrac{7}{4}, \tfrac{9}{4}\right) \Delta A$$

$$= \tfrac{77}{16}\left(\tfrac{3}{4}\right) + \tfrac{149}{16}\left(\tfrac{3}{4}\right) + \tfrac{101}{16}\left(\tfrac{3}{4}\right) + \tfrac{173}{16}\left(\tfrac{3}{4}\right) = \tfrac{375}{16} = 23.4375$$



5. The values of $f(x,y) = \sqrt{52 - x^2 - y^2}$ get smaller as we move farther from the origin, so on any of the subrectangles in the problem, the function will have its largest value at the lower left corner of the subrectangle and its smallest value at the upper right corner, and any other value will lie between these two. So using these subrectangles we have $U < V < L$. (Note that this is true no matter how R is divided into subrectangles.)

6. To approximate the volume, let R be the planar region corresponding to the surface of the water in the pool, and place R on coordinate axes so that x and y correspond to the dimensions given. Then we define $f(x,y)$ to be the depth of the water at (x,y), so the volume of water in the pool is the volume of the solid that lies above the rectangle $R = [0,20] \times [0,30]$ and below the graph of $f(x,y)$. We can estimate this volume using the Midpoint Rule with $m = 2$ and $n = 3$, so $\Delta A = 100$. Each subrectangle with its midpoint is shown in the figure. Then



$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{3} f\left(\overline{x}_i, \overline{y}_j\right) \Delta A = \Delta A[f(5,5) + f(5,15) + f(5,25) + f(15,5) + f(15,15) + f(15,25)]$$

$$= 100(3 + 7 + 10 + 3 + 5 + 8) = 3600$$

Thus, we estimate that the pool contains 3600 cubic feet of water.

Alternatively, we can approximate the volume with a Riemann sum where $m = 4$, $n = 6$ and the sample points are taken to be, for example, the upper right corner of each subrectangle. Then $\Delta A = 25$ and

$$V \approx \sum_{i=1}^{4} \sum_{j=1}^{6} f(x_i, y_j)\, \Delta A$$

$$= 25[3 + 4 + 7 + 8 + 10 + 8 + 4 + 6 + 8 + 10 + 12 + 10 + 3 + 4 + 5 + 6 + 8 + 7 + 2 + 2 + 2 + 3 + 4 + 4]$$

$$= 25(140) = 3500$$

So we estimate that the pool contains 3500 ft^3 of water.

7. (a) With $m = n = 2$, we have $\Delta A = 4$. Using the contour map to estimate the value of f at the center of each subrectangle, we have

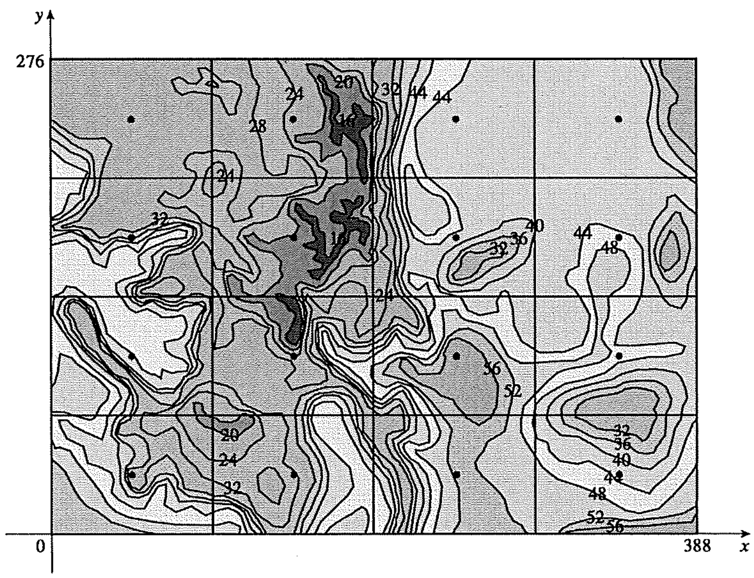
$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\overline{x}_i,\overline{y}_j)\,\Delta A = \Delta A[f(1,1)+f(1,3)+f(3,1)+f(3,3)] \approx 4(27+4+14+17) = 248$$

(b) $f_{\text{avg}} = \frac{1}{A(R)}\iint_R f(x,y)\,dA \approx \frac{1}{16}(248) = 15.5$

8. As in Example 9, we place the origin at the southwest corner of the state. Then $R = [0,388]\times[0,276]$ (in miles) is the rectangle corresponding to Colorado and we define $f(x,y)$ to be the temperature at the location (x,y). The average temperature is given by

$$f_{\text{avg}} = \frac{1}{A(R)}\iint_R f(x,y)\,dA = \frac{1}{388\cdot 276}\iint_R f(x,y)\,dA$$

To use the Midpoint Rule with $m = n = 4$, we divide R into 16 regions of equal size, as shown in the figure, with the center of each subrectangle indicated.



The area of each subrectangle is $\Delta A = \frac{388}{4}\cdot\frac{276}{4} = 6693$, so using the contour map to estimate the function values at each midpoint, we have

$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{4}\sum_{j=1}^{4} f(\overline{x}_i,\overline{y}_j)\,\Delta A\\ &\approx \Delta A\,[31+28+52+43+43+25+57+46+36+20+42+45+30+23+43+41]\\ &= 6693(605)\end{aligned}$$

Therefore, $f_{\text{avg}} \approx \dfrac{6693\cdot 605}{388\cdot 276} \approx 37.8$, so the average temperature in Colorado at 4:00 PM on a day in February was approximately 37.8°F.

9. $z = \sqrt{2} > 0$, so we can interpret the double integral as the volume of the solid S that lies below the plane $z = \sqrt{2}$ and above the rectangle $[2,6]\times[-1,5]$. S is a rectangular solid, so $\iint_R \sqrt{2}\,dA = 4\cdot 6\cdot\sqrt{2} = 24\sqrt{2}$.

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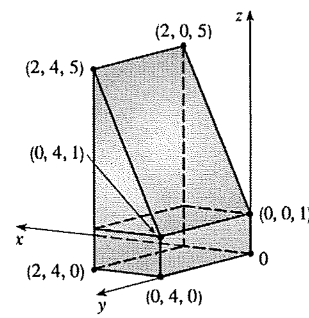
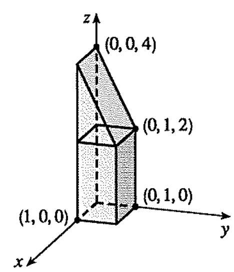
10. $z = 2x + 1 \geq 0$ for $0 \leq x \leq 2$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 2x + 1$ and above the rectangle $[0, 2] \times [0, 4]$. We can picture S as a rectangular solid (with height 1) surmounted by a triangular cylinder; thus

$$\iint_R (2x + 1)\,dA = (2)(4)(1) + \tfrac{1}{2}(2)(4)(4) = 24$$

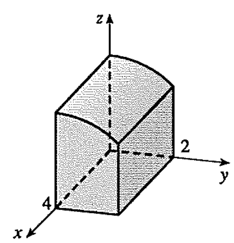


11. $z = 4 - 2y \geq 0$ for $0 \leq y \leq 1$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 4 - 2y$ and above the square $[0, 1] \times [0, 1]$. We can picture S as a rectangular solid (with height 2) surmounted by a triangular cylinder; thus

$$\iint_R (4 - 2y)\,dA = (1)(1)(2) + \tfrac{1}{2}(1)(1)(2) = 3$$



12. Here $z = \sqrt{9 - y^2}$, so $z^2 + y^2 = 9$, $z \geq 0$. Thus the integral represents the volume of the top half of the part of the circular cylinder $z^2 + y^2 = 9$ that lies above the rectangle $[0, 4] \times [0, 2]$.



13. $\int_0^2 (x + 3x^2y^2)\,dx = \left[\dfrac{x^2}{2} + 3\,\dfrac{x^3}{3}\,y^2\right]_{x=0}^{x=2} = \left[\tfrac{1}{2}x^2 + x^3y^2\right]_{x=0}^{x=2} = \left[\tfrac{1}{2}(2)^2 + (2)^3y^2\right] - \left[\tfrac{1}{2}(0)^2 + (0)^3y^2\right] = 2 + 8y^2,$

$\int_0^3 (x + 3x^2y^2)\,dy = \left[xy + 3x^2\,\dfrac{y^3}{3}\right]_{y=0}^{y=3} = \left[xy + x^2y^3\right]_{y=0}^{y=3} = \left[x(3) + x^2(3)^3\right] - \left[x(0) + x^2(0)^3\right] = 3x + 27x^2$

14. $\int_0^2 y\sqrt{x+2}\,dx = \left[y \cdot \tfrac{2}{3}(x+2)^{3/2}\right]_{x=0}^{x=2} = \tfrac{2}{3}y(4)^{3/2} - \tfrac{2}{3}y(2)^{3/2} = \tfrac{16}{3}y - \tfrac{4}{3}\sqrt{2}\,y = \tfrac{4}{3}(4 - \sqrt{2})\,y,$

$\int_0^3 y\sqrt{x+2}\,dy = \left[\dfrac{y^2}{2}\sqrt{x+2}\right]_{y=0}^{y=3} = \tfrac{1}{2}(3)^2\sqrt{x+2} - \tfrac{1}{2}(0)^2\sqrt{x+2} = \tfrac{9}{2}\sqrt{x+2}$

15. $\int_1^4 \int_0^2 (6x^2y - 2x)\,dy\,dx = \int_1^4 \left[3x^2y^2 - 2xy\right]_{y=0}^{y=2} dx = \int_1^4 \left[(12x^2 - 4x) - (0 - 0)\right] dx$

$= \int_1^4 (12x^2 - 4x)\,dx = \left[4x^3 - 2x^2\right]_1^4 = (256 - 32) - (4 - 2) = 222$

16. $\int_0^1 \int_0^1 (x+y)^2\,dx\,dy = \int_0^1 \int_0^1 (x^2 + 2xy + y^2)\,dx\,dy = \int_0^1 \left[\tfrac{1}{3}x^3 + x^2y + xy^2\right]_{x=0}^{x=1} dy$

$= \int_0^1 (\tfrac{1}{3} + y + y^2)\,dy = \left[\tfrac{1}{3}y + \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3\right]_0^1 = \tfrac{1}{3} + \tfrac{1}{2} + \tfrac{1}{3} - 0 = \tfrac{7}{6}$

17. $\int_0^1 \int_1^2 (x + e^{-y})\,dx\,dy = \int_0^1 \left[\frac{1}{2}x^2 + xe^{-y}\right]_{x=1}^{x=2} dy = \int_0^1 \left[(2 + 2e^{-y}) - (\frac{1}{2} + e^{-y})\right] dy$

$= \int_0^1 (\frac{3}{2} + e^{-y})\,dy = \left[\frac{3}{2}y - e^{-y}\right]_0^1 = (\frac{3}{2} - e^{-1}) - (0 - 1) = \frac{5}{2} - e^{-1}$

18. $\int_{-3}^1 \int_1^2 (x^2 + y^{-2})\,dy\,dx = \int_{-3}^1 \left[x^2 y - y^{-1}\right]_{y=1}^{y=2} dx = \int_{-3}^1 \left[(2x^2 - \frac{1}{2}) - (x^2 - 1)\right] dx$

$= \int_{-3}^1 (x^2 + \frac{1}{2})\,dx = \left[\frac{1}{3}x^3 + \frac{1}{2}x\right]_{-3}^1 = (\frac{1}{3} + \frac{1}{2}) - (-\frac{27}{3} - \frac{3}{2}) = \frac{34}{3}$

19. $\int_{-3}^3 \int_0^{\pi/2} (y + y^2 \cos x)\,dx\,dy = \int_{-3}^3 \left[xy + y^2 \sin x\right]_{x=0}^{x=\pi/2} dy = \int_{-3}^3 (\frac{\pi}{2}y + y^2)\,dy$

$= \left[\frac{\pi}{4}y^2 + \frac{1}{3}y^3\right]_{-3}^3 = \left[(\frac{9\pi}{4} + 9) - (\frac{9\pi}{4} - 9)\right] = 18$

20. $\displaystyle\int_1^3 \int_1^5 \frac{\ln y}{xy}\,dy\,dx = \int_1^3 \frac{1}{x}\,dx \int_1^5 \frac{\ln y}{y}\,dy$ [by Equation 11]

$\displaystyle= \Big[\ln|x|\Big]_1^3 \left[\tfrac{1}{2}(\ln y)^2\right]_1^5$ [substitute $u = \ln y \Rightarrow du = (1/y)\,dy$]

$= (\ln 3 - 0) \cdot \frac{1}{2}[(\ln 5)^2 - 0] = \frac{1}{2}(\ln 3)(\ln 5)^2$

21. $\displaystyle\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy\,dx = \int_1^4 \left[x \ln|y| + \frac{1}{x} \cdot \frac{1}{2}y^2\right]_{y=1}^{y=2} dx = \int_1^4 \left(x \ln 2 + \frac{3}{2x}\right) dx = \left[\tfrac{1}{2}x^2 \ln 2 + \tfrac{3}{2}\ln|x|\right]_1^4$

$= (8 \ln 2 + \frac{3}{2}\ln 4) - (\frac{1}{2}\ln 2 + 0) = \frac{15}{2}\ln 2 + \frac{3}{2}\ln 4$ or $\frac{15}{2}\ln 2 + 3\ln(4^{1/2}) = \frac{21}{2}\ln 2$

22. $\int_0^1 \int_0^2 ye^{x-y}\,dx\,dy = \int_0^1 \int_0^2 ye^x e^{-y}\,dx\,dy = \int_0^2 e^x\,dx \int_0^1 ye^{-y}\,dy$ [by Equation 11]

$= \left[e^x\right]_0^2 \left[(-y - 1)e^{-y}\right]_0^1$ [by integrating by parts]

$= (e^2 - e^0)\left[-2e^{-1} - (-e^0)\right] = (e^2 - 1)(1 - 2e^{-1})$ or $e^2 - 2e + 2e^{-1} - 1$

23. $\int_0^3 \int_0^{\pi/2} t^2 \sin^3\phi\,d\phi\,dt = \int_0^{\pi/2} \sin^3\phi\,d\phi \int_0^3 t^2\,dt$ [by Equation 11] $= \int_0^{\pi/2} (1 - \cos^2\phi)\sin\phi\,d\phi \int_0^3 t^2\,dt$

$= \left[\frac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi/2} \left[\frac{1}{3}t^3\right]_0^3 = \left[(0 - 0) - (\frac{1}{3} - 1)\right] \cdot \frac{1}{3}(27 - 0) = \frac{2}{3}(9) = 6$

24. $\int_0^1 \int_0^1 xy\sqrt{x^2 + y^2}\,dy\,dx = \int_0^1 x\left[\frac{1}{3}(x^2 + y^2)^{3/2}\right]_{y=0}^{y=1} dx = \frac{1}{3}\int_0^1 x[(x^2 + 1)^{3/2} - x^3]\,dx = \frac{1}{3}\int_0^1 [x(x^2 + 1)^{3/2} - x^4]\,dx$

$= \frac{1}{3}\left[\frac{1}{5}(x^2 + 1)^{5/2} - \frac{1}{5}x^5\right]_0^1 = \frac{1}{15}\left[(2^{5/2} - 1) - (1 - 0)\right] = \frac{2}{15}(2\sqrt{2} - 1)$

25. $\int_0^1 \int_0^1 v(u + v^2)^4\,du\,dv = \int_0^1 \left[\frac{1}{5}v(u + v^2)^5\right]_{u=0}^{u=1} dv = \frac{1}{5}\int_0^1 v\left[(1 + v^2)^5 - (0 + v^2)^5\right] dv$

$= \frac{1}{5}\int_0^1 \left[v(1 + v^2)^5 - v^{11}\right] dv = \frac{1}{5}\left[\frac{1}{2} \cdot \frac{1}{6}(1 + v^2)^6 - \frac{1}{12}v^{12}\right]_0^1$

[substitute $t = 1 + v^2 \Rightarrow dt = 2v\,dv$ in the first term]

$= \frac{1}{60}\left[(2^6 - 1) - (1 - 0)\right] = \frac{1}{60}(63 - 1) = \frac{31}{30}$

26. $\int_0^1 \int_0^1 \sqrt{s + t}\,ds\,dt = \int_0^1 \left[\frac{2}{3}(s + t)^{3/2}\right]_{s=0}^{s=1} dt = \frac{2}{3}\int_0^1 [(1 + t)^{3/2} - t^{3/2}]\,dt = \frac{2}{3}\left[\frac{2}{5}(1 + t)^{5/2} - \frac{2}{5}t^{5/2}\right]_0^1$

$= \frac{4}{15}[(2^{5/2} - 1) - (1 - 0)] = \frac{4}{15}(2^{5/2} - 2)$ or $\frac{8}{15}(2\sqrt{2} - 1)$

27. $\iint_R x\sec^2 y\,dA = \int_0^2 \int_0^{\pi/4} x\sec^2 y\,dy\,dx = \int_0^2 x\,dx \int_0^{\pi/4} \sec^2 y\,dy = \left[\frac{1}{2}x^2\right]_0^2 \Big[\tan y\Big]_0^{\pi/4}$

$= (2 - 0)(\tan\frac{\pi}{4} - \tan 0) = 2(1 - 0) = 2$

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28. $\iint_R (y + xy^{-2})\,dA = \int_1^2 \int_0^2 (y + xy^{-2})\,dx\,dy = \int_1^2 \left[xy + \frac{1}{2}x^2y^{-2}\right]_{x=0}^{x=2} dy = \int_1^2 \left(2y + 2y^{-2}\right) dy$
$= \left[y^2 - 2y^{-1}\right]_1^2 = (4 - 1) - (1 - 2) = 4$

29. $\displaystyle\iint_R \frac{xy^2}{x^2+1}\,dA = \int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\,dy\,dx = \int_0^1 \frac{x}{x^2+1}\,dx \int_{-3}^3 y^2\,dy = \left[\tfrac{1}{2}\ln(x^2+1)\right]_0^1 \left[\tfrac{1}{3}y^3\right]_{-3}^3$
$= \frac{1}{2}(\ln 2 - \ln 1) \cdot \frac{1}{3}(27 + 27) = 9\ln 2$

30. $\displaystyle\iint_R \frac{\tan\theta}{\sqrt{1-t^2}}\,dA = \int_0^{1/2} \int_0^{\pi/3} \frac{\tan\theta}{\sqrt{1-t^2}}\,d\theta\,dt = \int_0^{1/2} \frac{1}{\sqrt{1-t^2}}\,dt \int_0^{\pi/3} \tan\theta\,d\theta = \left[\sin^{-1} t\right]_0^{1/2} \left[\ln|\sec\theta|\right]_0^{\pi/3}$
$= \left(\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right)\left(\ln\left|\sec\frac{\pi}{3}\right| - \ln|\sec 0|\right) = \left(\frac{\pi}{6} - 0\right)(\ln 2 - \ln 1) = \frac{\pi}{6}\ln 2$

31. $\iint_R x\sin(x+y)\,dA = \int_0^{\pi/6}\int_0^{\pi/3} x\sin(x+y)\,dy\,dx$
$= \int_0^{\pi/6} \left[-x\cos(x+y)\right]_{y=0}^{y=\pi/3} dx = \int_0^{\pi/6} \left[x\cos x - x\cos\left(x + \frac{\pi}{3}\right)\right] dx$
$= x\left[\sin x - \sin\left(x + \frac{\pi}{3}\right)\right]_0^{\pi/6} - \int_0^{\pi/6} \left[\sin x - \sin\left(x + \frac{\pi}{3}\right)\right] dx$ [by integrating by parts separately for each term]
$= \frac{\pi}{6}\left[\frac{1}{2} - 1\right] - \left[-\cos x + \cos\left(x + \frac{\pi}{3}\right)\right]_0^{\pi/6} = -\frac{\pi}{12} - \left[-\frac{\sqrt{3}}{2} + 0 - \left(-1 + \frac{1}{2}\right)\right] = \frac{\sqrt{3}-1}{2} - \frac{\pi}{12}$

32. $\displaystyle\iint_R \frac{x}{1+xy}\,dA = \int_0^1 \int_0^1 \frac{x}{1+xy}\,dy\,dx = \int_0^1 \left[\ln(1+xy)\right]_{y=0}^{y=1} dx = \int_0^1 \left[\ln(1+x) - \ln 1\right] dx$
$= \int_0^1 \ln(1+x)\,dx = \left[(1+x)\ln(1+x) - x\right]_0^1$ [by integrating by parts]
$= (2\ln 2 - 1) - (\ln 1 - 0) = 2\ln 2 - 1$

33. $\iint_R ye^{-xy}\,dA = \int_0^3 \int_0^2 ye^{-xy}\,dx\,dy = \int_0^3 \left[-e^{-xy}\right]_{x=0}^{x=2} dy = \int_0^3 (-e^{-2y} + 1)\,dy = \left[\frac{1}{2}e^{-2y} + y\right]_0^3$
$= \frac{1}{2}e^{-6} + 3 - \left(\frac{1}{2} + 0\right) = \frac{1}{2}e^{-6} + \frac{5}{2}$

34. $\displaystyle\iint_R \frac{1}{1+x+y}\,dA = \int_1^3 \int_1^2 \frac{1}{1+x+y}\,dy\,dx = \int_1^3 \left[\ln(1+x+y)\right]_{y=1}^{y=2} dx = \int_1^3 \left[\ln(x+3) - \ln(x+2)\right] dx$
$= \left[\left((x+3)\ln(x+3) - (x+3)\right) - \left((x+2)\ln(x+2) - (x+2)\right)\right]_1^3$
[by integrating by parts separately for each term]
$= (6\ln 6 - 6 - 5\ln 5 + 5) - (4\ln 4 - 4 - 3\ln 3 + 3) = 6\ln 6 - 5\ln 5 - 4\ln 4 + 3\ln 3$

35. $z = f(x,y) = 4 - x - 2y \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So the solid is the region in the first octant which lies below the plane $z = 4 - x - 2y$ and above $[0,1] \times [0,1]$.