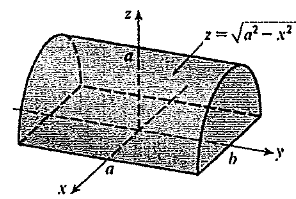
HW #13, Section 15.3 SOLUTIONS

$\iint_D \sqrt{a^2 - x^2}\, dA$ represents the volume of the solid region under the graph of $z = \sqrt{a^2 - x^2}$ and above the rectangle D, namely a half circular cylinder with radius a and length $2b$ (see the figure) whose volume is $\frac{1}{2} \cdot \pi r^2 h = \frac{1}{2}\pi a^2 (2b) = \pi a^2 b$. Thus



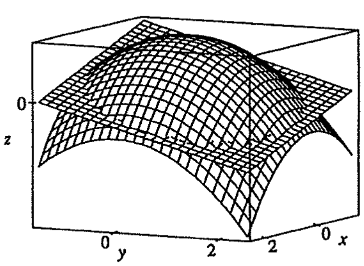
$\iint_D \left(ax^3 + by^3 + \sqrt{a^2 - x^2}\right) dA = 0 + 0 + \pi a^2 b = \pi a^2 b$.

80. By the Extreme Value Theorem (14.7.8), f has an absolute minimum value m and an absolute maximum value M in D. Then by Property 15.2.10, $mA(D) \le \iint_D f(x,y)\, dA \le MA(D)$. Dividing through by the positive number $A(D)$, we get $m \le \dfrac{1}{A(D)} \iint_D f(x,y)\, dA \le M$. This says that the average value of f over D lies between m and M. But f is continuous on D and takes on the values m and M, and so by the Intermediate Value Theorem must take on all values between m and M. Specifically, there exists a point (x_0, y_0) in D such that $f(x_0, y_0) = \dfrac{1}{A(D)} \iint_D f(x,y)\, dA$ or equivalently $\iint_D f(x,y)\, dA = f(x_0, y_0)\, A(D)$.

81. For each r such that D_r lies within the domain, $A(D_r) = \pi r^2$, and by the Mean Value Theorem for double integrals there exists (x_r, y_r) in D_r such that $f(x_r, y_r) = \dfrac{1}{\pi r^2} \iint_{D_r} f(x,y)\, dA$. But $\lim\limits_{r \to 0^+} (x_r, y_r) = (a, b)$,

so $\lim\limits_{r \to 0^+} \dfrac{1}{\pi r^2} \iint_{D_r} f(x,y)\, dA = \lim\limits_{r \to 0^+} f(x_r, y_r) = f(a,b)$ by the continuity of f.

82. To find the equations of the boundary curves, we require that the z-values of the two surfaces be the same. In Maple, we use the command `solve(4-x^2-y^2=1-x-y,y);` and in Mathematica, we use `Solve[4-x^2-y^2==1-x-y,y]`. We find that the curves have equations $y = \dfrac{1 \pm \sqrt{13 + 4x - 4x^2}}{2}$. To find the two points of intersection of these curves, we use the CAS to solve $13 + 4x - 4x^2 = 0$, finding that $x = \frac{1 \pm \sqrt{14}}{2}$. So, using the CAS to evaluate the integral, the volume of intersection is



$$V = \int_{(1-\sqrt{14})/2}^{(1+\sqrt{14})/2} \int_{\left(1-\sqrt{13+4x-4x^2}\right)/2}^{\left(1+\sqrt{13+4x-4x^2}\right)/2} \left[(4 - x^2 - y^2) - (1 - x - y)\right] dy\, dx = \frac{49\pi}{8}$$

15.3 Double Integrals in Polar Coordinates

1. The region R is more easily described with polar coordinates: $R = \{(r, \theta) \mid 0 \le r \le 4,\ 0 \le \theta \le 3\pi/2\}$. Thus, $\iint_R f(x,y)\, dA = \int_0^{3\pi/2} \int_0^4 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

2. The region R is more easily described by rectangular coordinates: $R = \{(x,y) \mid -1 \le x \le 1,\ -x \le y \le 1\}$. Thus, $\iint_R f(x,y)\, dA = \int_{-1}^{1} \int_{-x}^{1} f(x,y)\, dy\, dx$.

3. The region R is more easily described with polar coordinates: $R = \{(r, \theta) \mid 1 \le r \le 3, 0 \le \theta \le \pi\}$.

Thus, $\iint_R f(x, y)\, dA = \int_0^{\pi} \int_1^3 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

4. The region R is more easily described by polar coordinates: $R = \left\{(r, \theta) \mid 0 \le r \le 3, -\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\right\}$.

Thus, $\iint_R f(x, y)\, dA = \int_{-\pi/4}^{3\pi/4} \int_0^3 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

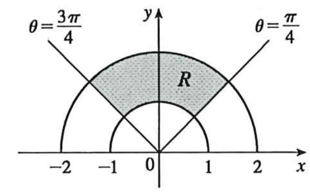
5. The region R is more easily described with rectangular coordinates: $R = \{(x, y) \mid 2y - 2 \le x \le -2y + 2, 0 \le y \le 1\}$.

Thus, $\iint_R f(x, y)\, dA = \int_0^1 \int_{2y-2}^{-2y+2} f(x, y)\, dx\, dy$.

6. The region R is more easily described with polar coordinates: $R = \{(r, \theta) \mid 8 \le r \le 10, 0 \le \theta \le 2\pi\}$.

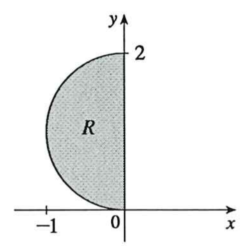
Thus, $\iint_R f(x, y)\, dA = \int_0^{2\pi} \int_8^{10} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

7. The integral $\int_{\pi/4}^{3\pi/4} \int_1^2 r\, dr\, d\theta$ represents the area of the region $R = \{(r, \theta) \mid 1 \le r \le 2, \pi/4 \le \theta \le 3\pi/4\}$, the top quarter portion of a ring (annulus).



$$\int_{\pi/4}^{3\pi/4} \int_1^2 r\, dr\, d\theta = \left(\int_{\pi/4}^{3\pi/4} d\theta\right)\left(\int_1^2 r\, dr\right)$$
$$= \left[\theta\right]_{\pi/4}^{3\pi/4} \left[\tfrac{1}{2}r^2\right]_1^2 = \left(\tfrac{3\pi}{4} - \tfrac{\pi}{4}\right) \cdot \tfrac{1}{2}(4 - 1) = \tfrac{\pi}{2} \cdot \tfrac{3}{2} = \tfrac{3\pi}{4}$$

8. The integral $\int_{\pi/2}^{\pi} \int_0^{2\sin\theta} r\, dr\, d\theta$ represents the area of the region $R = \{(r, \theta) \mid 0 \le r \le 2\sin\theta, \pi/2 \le \theta \le \pi\}$. Since $r = 2\sin\theta \;\Rightarrow\; r^2 = 2r\sin\theta \;\Leftrightarrow\; x^2 + y^2 = 2y \;\Leftrightarrow\; x^2 + (y - 1)^2 = 1$, R is the portion in the second quadrant of a disk of radius 1 with center $(0, 1)$.

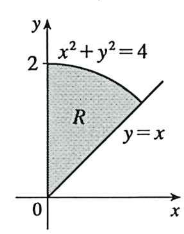


$$\int_{\pi/2}^{\pi} \int_0^{2\sin\theta} r\, dr\, d\theta = \int_{\pi/2}^{\pi} \left[\tfrac{1}{2}r^2\right]_{r=0}^{r=2\sin\theta} d\theta = \int_{\pi/2}^{\pi} 2\sin^2\theta\, d\theta$$
$$= \int_{\pi/2}^{\pi} 2 \cdot \tfrac{1}{2}(1 - \cos 2\theta)\, d\theta = \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/2}^{\pi}$$
$$= \pi - 0 - \tfrac{\pi}{2} + 0 = \tfrac{\pi}{2}$$

9. The half-disk D can be described in polar coordinates as $D = \{(r, \theta) \mid 0 \le r \le 5, 0 \le \theta \le \pi\}$. Then

$$\iint_D x^2 y\, dA = \int_0^{\pi} \int_0^5 (r\cos\theta)^2 (r\sin\theta)\, r\, dr\, d\theta = \left(\int_0^{\pi} \cos^2\theta \sin\theta\, d\theta\right)\left(\int_0^5 r^4\, dr\right)$$
$$= \left[-\tfrac{1}{3}\cos^3\theta\right]_0^{\pi} \left[\tfrac{1}{5}r^5\right]_0^5 = -\tfrac{1}{3}(-1 - 1) \cdot 625 = \tfrac{1250}{3}$$

10. The region R is $\frac{1}{8}$ of a disk, as shown in the figure, and can be described by $R = \{(r, \theta) \mid 0 \le r \le 2, \pi/4 \le \theta \le \pi/2\}$. Thus



$$\iint_R (2x - y)\, dA = \int_{\pi/4}^{\pi/2} \int_0^2 (2r\cos\theta - r\sin\theta)\, r\, dr\, d\theta$$
$$= \int_{\pi/4}^{\pi/2} (2\cos\theta - \sin\theta)\, d\theta \int_0^2 r^2\, dr$$
$$= \left[2\sin\theta + \cos\theta\right]_{\pi/4}^{\pi/2} \left[\tfrac{1}{3}r^3\right]_0^2$$
$$= \left(2 + 0 - \sqrt{2} - \tfrac{\sqrt{2}}{2}\right)\left(\tfrac{8}{3}\right) = \tfrac{16}{3} - 4\sqrt{2}$$

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11. $\iint_R \sin(x^2+y^2)\,dA = \int_0^{\pi/2}\int_1^3 \sin(r^2)\,r\,dr\,d\theta = \int_0^{\pi/2} d\theta \int_1^3 r\sin(r^2)\,dr = [\theta]_0^{\pi/2}\left[-\frac{1}{2}\cos(r^2)\right]_1^3$

$= \left(\frac{\pi}{2}\right)\left[-\frac{1}{2}(\cos 9 - \cos 1)\right] = \frac{\pi}{4}(\cos 1 - \cos 9)$

12. $\displaystyle\iint_R \frac{y^2}{x^2+y^2}\,dA = \int_0^{2\pi}\int_a^b \frac{(r\sin\theta)^2}{r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\sin^2\theta\,d\theta\int_a^b r\,dr = \int_0^{2\pi}\frac{1}{2}(1-\cos 2\theta)\,d\theta\int_a^b r\,dr$

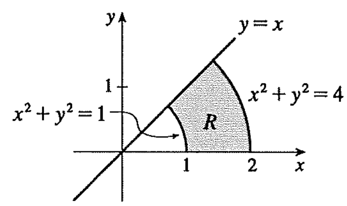
$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi}\left[\frac{1}{2}r^2\right]_a^b = \frac{1}{2}(2\pi - 0 - 0)\cdot\frac{1}{2}(b^2-a^2) = \frac{\pi}{2}(b^2-a^2)$

13. $\iint_D e^{-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2}\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} d\theta\int_0^2 re^{-r^2}\,dr$

$= [\theta]_{-\pi/2}^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\frac{1}{2}\right)(e^{-4}-e^0) = \frac{\pi}{2}(1-e^{-4})$

14. $\iint_D \cos\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^2 \cos\sqrt{r^2}\;r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^2 r\cos r\,dr$. For the second integral, integrate by parts with $u = r$, $dv = \cos r\,dr$. Then $\iint_D \cos\sqrt{x^2+y^2}\,dA = [\theta]_0^{2\pi}[r\sin r + \cos r]_0^2 = 2\pi(2\sin 2 + \cos 2 - 1)$.

15. R is the region shown in the figure, and can be described by $R = \{(r,\theta) \mid 0 \le \theta \le \pi/4, 1 \le r \le 2\}$. Thus $\iint_R \arctan(y/x)\,dA = \int_0^{\pi/4}\int_1^2 \arctan(\tan\theta)\,r\,dr\,d\theta$ since $y/x = \tan\theta$. Also, $\arctan(\tan\theta) = \theta$ for $0 \le \theta \le \pi/4$, so the integral becomes $\int_0^{\pi/4}\int_1^2 \theta\,r\,dr\,d\theta = \int_0^{\pi/4}\theta\,d\theta\int_1^2 r\,dr = \left[\frac{1}{2}\theta^2\right]_0^{\pi/4}\left[\frac{1}{2}r^2\right]_1^2 = \frac{\pi^2}{32}\cdot\frac{3}{2} = \frac{3}{64}\pi^2$.



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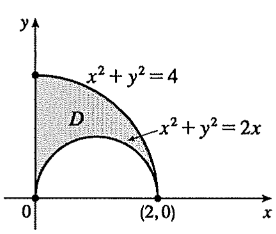


$$\iint_D x\,dA = \iint_{\substack{x^2+y^2\le 4\\ x\ge 0,\,y\ge 0}} x\,dA - \iint_{\substack{(x-1)^2+y^2\le 1\\ y\ge 0}} x\,dA$$

$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\,dr\,d\theta$

$= \int_0^{\pi/2}\frac{1}{3}(8\cos\theta)\,d\theta - \int_0^{\pi/2}\frac{1}{3}(8\cos^4\theta)\,d\theta$

$= \frac{8}{3}[\sin\theta]_0^{\pi/2} - \frac{8}{12}\left[\cos^3\theta\sin\theta + \frac{3}{2}(\theta + \sin\theta\cos\theta)\right]_0^{\pi/2}$

$= \frac{8}{3} - \frac{2}{3}\left[0 + \frac{3}{2}\left(\frac{\pi}{2}\right)\right] = \frac{16-3\pi}{6}$

17. By symmetry, the area of the region is 4 times the area of the region D in the first quadrant enclosed by the cardiod $r = 1-\cos\theta$ (see the figure). Here $D = \{(r,\theta) \mid 0 \le r \le 1-\cos\theta, 0 \le \theta \le \pi/2\}$, so the total area is

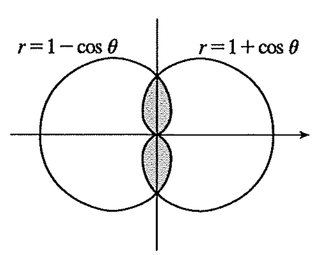
$4A(D) = 4\iint_D dA = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta = 4\int_0^{\pi/2}\left[\frac{1}{2}r^2\right]_{r=0}^{r=1-\cos\theta}d\theta$

$= 2\int_0^{\pi/2}(1-\cos\theta)^2 d\theta = 2\int_0^{\pi/2}(1-2\cos\theta+\cos^2\theta)\,d\theta$

$= 2\int_0^{\pi/2}\left[1 - 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right]d\theta$

$= 2\left[\theta - 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2}$

$= 2\left(\frac{\pi}{2} - 2 + \frac{\pi}{4}\right) = \frac{3\pi}{2} - 4$



18. The region D is described by $D = \{(r, \theta) \mid 0 \le r \le \sqrt{\theta}, 0 \le \theta \le 2\pi\}$, so the area is

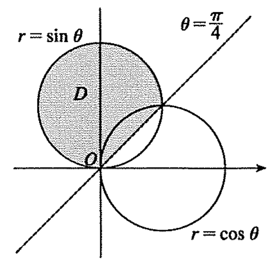
$$A(D) = \int_0^{2\pi} \int_0^{\sqrt{\theta}} r\,dr\,d\theta = \int_0^{2\pi} \left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{\theta}} d\theta = \int_0^{2\pi} \frac{\theta}{2}\,d\theta = \left[\frac{\theta^2}{4}\right]_0^{2\pi} = \pi^2.$$

19. By symmetry, the total area is twice the area defined by

$D = \{(r, \theta) \mid 0 \le r \le \sin\theta, \pi/4 \le \theta \le \pi\}$ (see the figure).

The total area is

$$2A(D) = 2\int_{\pi/4}^{\pi} \int_0^{\sin\theta} r\,dr\,d\theta = 2 \cdot \tfrac{1}{2} \int_{\pi/4}^{\pi} \left[r^2\right]_{r=0}^{r=\sin\theta} d\theta = \int_{\pi/4}^{\pi} \sin^2\theta\,d\theta$$
$$= \int_{\pi/4}^{\pi} \tfrac{1}{2}(1 - \cos 2\theta)\,d\theta = \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi}$$
$$= \tfrac{1}{2}(\pi - 0) - \tfrac{1}{2}\left(\tfrac{\pi}{4} - \tfrac{1}{2}\right) = \tfrac{3\pi}{8} + \tfrac{1}{4}$$



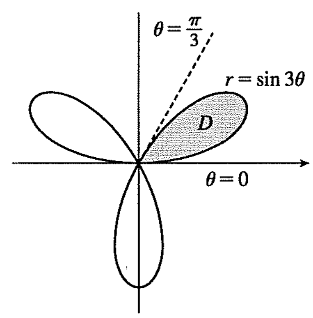
20. By symmetry, the area of the region is 4 times the area of the region D in the first quadrant between the circle $r = 1/\sqrt{2}$ and the curve $r^2 = \cos 2\theta \;\Rightarrow\; r = \sqrt{\cos 2\theta}$. The curves intersect in the first quadrant when $\cos 2\theta = \left(\frac{1}{\sqrt{2}}\right)^2 \;\Rightarrow\; \cos 2\theta = \frac{1}{2} \;\Rightarrow\; 2\theta = \frac{\pi}{3} \;\Rightarrow\; \theta = \frac{\pi}{6}$. Thus, $D = \{(r, \theta) \mid 1/\sqrt{2} \le r \le \sqrt{\cos 2\theta}, 0 \le \theta \le \pi/6\}$, so the total area is

$$4A(D) = 4\int_0^{\pi/6} \int_{1/\sqrt{2}}^{\sqrt{\cos 2\theta}} r\,dr\,d\theta = 4 \cdot \frac{1}{2} \int_0^{\pi/6} \left[r^2\right]_{r=1/\sqrt{2}}^{r=\sqrt{\cos 2\theta}} d\theta = 2\int_0^{\pi/6} \left[\cos 2\theta - \frac{1}{2}\right] d\theta$$
$$= 2\left[\tfrac{1}{2}\sin 2\theta - \tfrac{\theta}{2}\right]_0^{\pi/6} = \tfrac{\sqrt{3}}{2} - \tfrac{\pi}{6}$$

21. One loop is given by the region

$D = \{(r, \theta) \mid 0 \le r \le \sin 3\theta, 0 \le \theta \le \pi/3\}$, so the area is

$$\iint_D dA = \int_0^{\pi/3} \int_0^{\sin 3\theta} r\,dr\,d\theta = \tfrac{1}{2} \int_0^{\pi/3} \left[r^2\right]_{r=0}^{r=\sin 3\theta} d\theta$$
$$= \tfrac{1}{2} \int_0^{\pi/3} \sin^2 3\theta\,d\theta = \tfrac{1}{2} \int_0^{\pi/3} \tfrac{1}{2}(1 - \cos 6\theta)\,d\theta$$
$$= \tfrac{1}{4}\left[\theta - \tfrac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \tfrac{\pi}{12}$$



22. In polar coordinates the circle $(x-1)^2 + y^2 = 1 \;\Leftrightarrow\; x^2 + y^2 = 2x$ is $r^2 = 2r\cos\theta \;\Rightarrow\; r = 2\cos\theta$, and the circle $x^2 + y^2 = 1$ is $r = 1$. The curves intersect in the first quadrant when $2\cos\theta = 1 \;\Rightarrow\; \cos\theta = \frac{1}{2} \;\Rightarrow\; \theta = \pi/3$, so the portion of the region in the first quadrant is given by $D = \{(r, \theta) \mid 1 \le r \le 2\cos\theta, 0 \le \theta \le \pi/3\}$. By symmetry, the total area is twice the area of D:

$$2A(D) = 2\iint_D dA = 2\int_0^{\pi/3} \int_1^{2\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/3} \left[\tfrac{1}{2}r^2\right]_{r=1}^{r=2\cos\theta} d\theta$$
$$= \int_0^{\pi/3} \left(4\cos^2\theta - 1\right) d\theta = \int_0^{\pi/3} \left[4 \cdot \tfrac{1}{2}(1 + \cos 2\theta) - 1\right] d\theta$$
$$= \int_0^{\pi/3} (1 + 2\cos 2\theta)\,d\theta = \left[\theta + \sin 2\theta\right]_0^{\pi/3} = \tfrac{\pi}{3} + \tfrac{\sqrt{3}}{2}$$