Sec 7.3

7.3 Trigonometric Substitution

1. (a) Use $x = \tan\theta$, where $-\pi/2 < \theta < \pi/2$, since the integrand contains the expression $\sqrt{1+x^2}$.

(b) $x = \tan\theta \quad\Rightarrow\quad dx = \sec^2\theta\, d\theta$ and $\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$.

Then $\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int \frac{\tan^3\theta}{\sec\theta}\sec^2\theta\,d\theta = \int \tan^3\theta\sec\theta\,d\theta.$

2. (a) Use $x = 3\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$, since the integrand contains the expression $\sqrt{3^2 - x^2}$.

(b) $x = 3\sin\theta \quad\Rightarrow\quad dx = 3\cos\theta\,d\theta$ and

$$\begin{aligned}\sqrt{9-x^2} &= \sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta}\\ &= 3\,|\cos\theta| = 3\cos\theta\end{aligned}$$

Then $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\,dx = \int \frac{27\sin^3\theta}{3\cos\theta}\,3\cos\theta\,d\theta = \int 27\sin^3\theta\,d\theta.$

3. (a) Use $x = \sqrt{2}\sec\theta$, where $0 < \theta < \pi/2$ or $\pi < \theta < 3\pi/2$, since the integrand contains the expression $\sqrt{x^2-2}$.

(b) $x = \sqrt{2}\sec\theta \quad\Rightarrow\quad dx = \sqrt{2}\sec\theta\tan\theta\,d\theta$ and

$\sqrt{x^2-2} = \sqrt{2\sec^2\theta - 2} = \sqrt{2(\sec^2\theta - 1)} = \sqrt{2\tan^2\theta} = \sqrt{2}\,|\tan\theta| = \sqrt{2}\tan\theta.$

Then $\displaystyle\int \frac{x^2}{\sqrt{x^2-2}}\,dx = \int \frac{2\sec^2\theta}{\sqrt{2}\tan\theta}\sqrt{2}\sec\theta\tan\theta\,d\theta = \int 2\sec^3\theta\,d\theta.$

4. (a) Use $x = \frac{3}{2}\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$, since the integrand contains the expression

$(9-4x^2)^{3/2} = 4^{3/2}\left(\frac{9}{4} - x^2\right)^{3/2}.$

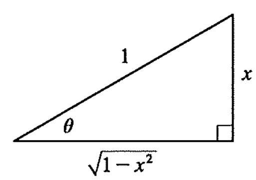
(b) $x = \frac{3}{2}\sin\theta \quad\Rightarrow\quad dx = \frac{3}{2}\cos\theta\,d\theta$ and

$(9-4x^2)^{3/2} = \left(\sqrt{9-9\sin^2\theta}\right)^3 = \left(\sqrt{9(1-\sin^2\theta)}\right)^3 = \left(\sqrt{9\cos^2\theta}\right)^3 = \left(3\,|\cos\theta|\right)^3 = 27\cos^3\theta.$

Then $\displaystyle\int \frac{x^3}{(9-4x^2)^{3/2}}\,dx = \int \frac{\frac{27}{8}\sin^3\theta}{27\cos^3\theta}\left(\tfrac{3}{2}\cos\theta\,d\theta\right) = \int \tfrac{3}{16}\sin^3\theta\sec^2\theta\,d\theta.$

5. Let $x = \sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = \cos\theta\,d\theta$ and

$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$. Thus,



$$\begin{aligned}\int \frac{x^3}{\sqrt{1-x^2}}\,dx &= \int \frac{\sin^3\theta}{\cos\theta}\cos\theta\;d\theta = \int (1-\cos^2\theta)\sin\theta\,d\theta\\ &\stackrel{c}{=} \int (1-u^2)(-du) = \int (-1+u^2)\,du = -u + \frac{1}{3}u^3 + C\\ &= -\cos\theta + \tfrac{1}{3}\cos^3\theta + C = -\sqrt{1-x^2} + \tfrac{1}{3}\left(\sqrt{1-x^2}\right)^3 + C\end{aligned}$$

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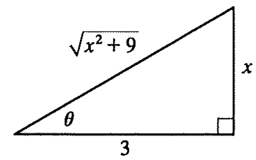
6. Let $x = 3\tan\theta$, where $-\pi/2 < \theta < \pi/2$. Then $dx = 3\sec^2\theta\,d\theta$ and

$\sqrt{9+x^2} = \sqrt{9+9\tan^2\theta} = \sqrt{9\,(1+\tan^2\theta)} = \sqrt{9\sec^2\theta} = 3\,|\sec\theta| = 3\sec\theta$. Thus,

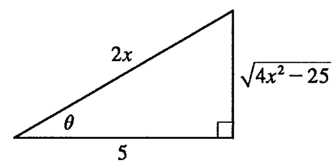


$$\begin{aligned}
\int \frac{x^3}{\sqrt{9+x^2}}\,dx &= \int \frac{27\tan^3\theta}{3\sec\theta}\left(3\sec^2\theta\,d\theta\right) = \int 27\tan^3\theta\sec\theta\,d\theta \\
&= 27\int\left(\sec^2\theta-1\right)\tan\theta\sec\theta\,d\theta \\
&= 27\int(u^2-1)\,du \qquad [u=\sec\theta,\ du=\sec\theta\tan\theta\,d\theta] \\
&= 27\left(\tfrac{1}{3}u^3-u\right)+C = 9\sec^3\theta - 27\sec\theta + C \\
&= 9\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - \frac{27\sqrt{x^2+9}}{3} + C = \frac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9} + C
\end{aligned}$$

7. Let $x = \frac{5}{2}\sec\theta$, where $0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = \frac{5}{2}\sec\theta\tan\theta\,d\theta$

and $\sqrt{4x^2-25} = \sqrt{25\sec^2\theta - 25} = \sqrt{25\tan^2\theta} = 5\,|\tan\theta| = 5\tan\theta$ for

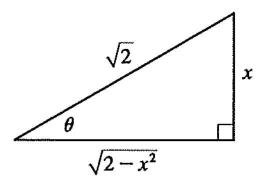
the relevant values of θ, so



$$\begin{aligned}
\int \frac{\sqrt{4x^2-25}}{x}\,dx &= \int \frac{5\tan\theta}{\frac{5}{2}\sec\theta}\left(\frac{5}{2}\sec\theta\tan\theta\,d\theta\right) = 5\int\tan^2\theta\,d\theta \\
&= 5(\tan\theta-\theta)+C \qquad [\text{by Exercise 7.1.59 or integration by parts}] \\
&= 5\left(\frac{\sqrt{4x^2-25}}{5} - \sec^{-1}\left(\frac{2x}{5}\right)\right) + C = \sqrt{4x^2-25} - 5\sec^{-1}\left(\frac{2}{5}x\right) + C
\end{aligned}$$

8. Let $x = \sqrt{2}\,\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = \sqrt{2}\cos\theta\,d\theta$ and

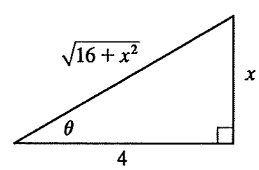
$\sqrt{2-x^2} = \sqrt{2-2\sin^2\theta} = \sqrt{2\cos^2\theta} = \sqrt{2}\,|\cos\theta| = \sqrt{2}\cos\theta$. Thus,



$$\begin{aligned}
\int \frac{\sqrt{2-x^2}}{x^2}\,dx &= \int \frac{\sqrt{2}\cos\theta}{2\sin^2\theta}\sqrt{2}\,\cos\theta\,d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}d\theta = \int\cot^2\theta\,d\theta \\
&= \int(\csc^2\theta-1)\,d\theta = -\cot\theta - \theta + C \\
&= -\frac{\sqrt{2-x^2}}{x} - \sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + C
\end{aligned}$$

9. Let $x = 4\tan\theta$, where $-\pi/2 < \theta < \pi/2$. Then $dx = 4\sec^2\theta\,d\theta$ and

$\sqrt{16+x^2} = \sqrt{16+16\tan^2\theta} = \sqrt{16\sec^2\theta} = 4\,|\sec\theta| = 4\sec\theta$. Thus,



$$\begin{aligned}
\int x^3\sqrt{16+x^2}\,dx &= \int 64\tan^3\theta\,(4\sec\theta)(4\sec^2\theta\,d\theta) = 1024\int\tan^3\theta\sec^3\theta\,d\theta \\
&= 1024\int\tan^2\theta\sec^2\theta\sec\theta\tan\theta\,d\theta = 1024\int(\sec^2\theta-1)\,\sec^2\theta\sec\theta\,\tan\theta\,d\theta \\
&= 1024\int(u^2-1)u^2\,du \qquad [u=\sec\theta,\ du=\sec\theta\,\tan\theta\,d\theta] \\
&= 1024\left(\frac{1}{5}u^5 - \frac{1}{3}u^3\right) + C = \frac{1024}{5}\left(\frac{\sqrt{16+x^2}}{4}\right)^5 - \frac{1024}{3}\left(\frac{\sqrt{16+x^2}}{4}\right)^3 + C \\
&= \left(16+x^2\right)^{3/2}\left(\frac{1}{5}\left(16+x^2\right) - \frac{16}{3}\right) + C = \left(16+x^2\right)^{3/2}\left(\frac{1}{5}x^2 - \frac{32}{15}\right) + C
\end{aligned}$$

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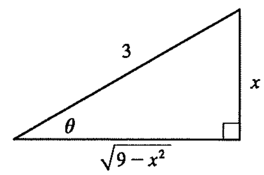
10. Let $x = 3\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 3\cos\theta\,d\theta$ and $\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\,|\cos\theta| = 3\cos\theta$.

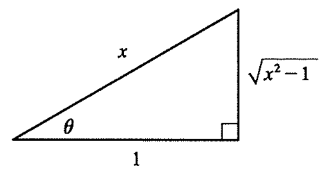


$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta}{3\cos\theta}\,3\cos\theta\,d\theta = 9\int \sin^2\theta\,d\theta$$

$$= 9\int \tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{9}{2}\left(\theta - \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{9}{2}\theta - \tfrac{9}{4}(2\sin\theta\,\cos\theta) + C$$

$$= \frac{9}{2}\sin^{-1}\!\left(\frac{x}{3}\right) - \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C = \frac{9}{2}\sin^{-1}\!\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C$$

11. Let $x = \sec\theta$, where $0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = \sec\theta\,\tan\theta\,d\theta$ and $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$ for the relevant values of θ, so



$$\int \frac{\sqrt{x^2-1}}{x^4}\,dx = \int \frac{\tan\theta}{\sec^4\theta}\,\sec\theta\,\tan\theta\,d\theta = \int \tan^2\theta\,\cos^3\theta\,d\theta$$

$$= \int \sin^2\theta\,\cos\theta\,d\theta \overset{s}{=} \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\sin^3\theta + C$$

$$= \frac{1}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C = \frac{1}{3}\frac{(x^2-1)^{3/2}}{x^3} + C$$

12. Let $u = 36 - x^2$, so $du = -2x\,dx$. When $x = 0$, $u = 36$; when $x = 3$, $u = 27$. Thus,

$$\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx = \int_{36}^{27} \frac{1}{\sqrt{u}}\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\Big[2\sqrt{u}\Big]_{36}^{27} = -\left(\sqrt{27}-\sqrt{36}\right) = 6 - 3\sqrt{3}$$

Another method: Let $x = 6\sin\theta$, so $dx = 6\cos\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 3 \;\Rightarrow\; \theta = \frac{\pi}{6}$. Then

$$\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx = \int_0^{\pi/6} \frac{6\sin\theta}{\sqrt{36(1-\sin^2\theta)}}\,6\cos\theta\,d\theta = \int_0^{\pi/6} \frac{6\sin\theta}{6\cos\theta}\,6\cos\theta\,d\theta = 6\int_0^{\pi/6} \sin\theta\,d\theta$$

$$= 6\Big[-\cos\theta\Big]_0^{\pi/6} = 6\left(-\tfrac{\sqrt{3}}{2} + 1\right) = 6 - 3\sqrt{3}$$

13. Let $x = a\tan\theta$, where $a > 0$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = a\,\sec^2\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = a \;\Rightarrow\; \theta = \frac{\pi}{4}$. Thus,

$$\int_0^a \frac{dx}{(a^2+x^2)^{3/2}} = \int_0^{\pi/4} \frac{a\sec^2\theta\,d\theta}{[a^2(1+\tan^2\theta)]^{3/2}} = \int_0^{\pi/4} \frac{a\sec^2\theta\,d\theta}{a^3\sec^3\theta} = \frac{1}{a^2}\int_0^{\pi/4} \cos\theta\,d\theta$$

$$= \frac{1}{a^2}\Big[\sin\theta\Big]_0^{\pi/4} = \frac{1}{a^2}\left(\frac{\sqrt{2}}{2} - 0\right) = \frac{1}{\sqrt{2}\,a^2}.$$

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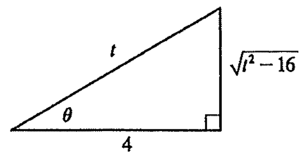
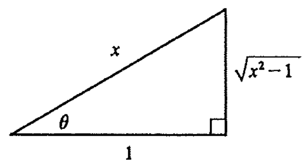
14. Let $t = 4\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dt = 4\sec\theta\,\tan\theta\,d\theta$ and $\sqrt{t^2-16} = \sqrt{16\sec^2\theta - 16} = \sqrt{16\tan^2\theta} = 4\tan\theta$ for the relevant values of θ, so



$$\int \frac{dt}{t^2\sqrt{t^2-16}} = \int \frac{4\sec\theta\,\tan\theta\,d\theta}{16\sec^2\theta\cdot 4\tan\theta} = \frac{1}{16}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{16}\int \cos\theta\,d\theta$$

$$= \frac{1}{16}\sin\theta + C = \frac{1}{16}\frac{\sqrt{t^2-16}}{t} + C = \frac{\sqrt{t^2-16}}{16t} + C$$

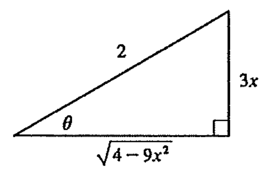
15. Let $x = \sec\theta$, so $dx = \sec\theta\,\tan\theta\,d\theta$, $x = 2 \;\Rightarrow\; \theta = \frac{\pi}{3}$, and $x = 3 \;\Rightarrow\; \theta = \sec^{-1}3$. Then



$$\int_2^3 \frac{dx}{(x^2-1)^{3/2}} = \int_{\pi/3}^{\sec^{-1}3} \frac{\sec\theta\,\tan\theta\,d\theta}{\tan^3\theta} = \int_{\pi/3}^{\sec^{-1}3} \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$\overset{s}{=} \int_{\sqrt{3}/2}^{\sqrt{8}/3} \frac{1}{u^2}\,du = \left[-\frac{1}{u}\right]_{\sqrt{3}/2}^{\sqrt{8}/3} = \frac{-3}{\sqrt{8}} + \frac{2}{\sqrt{3}} = -\frac{3}{4}\sqrt{2} + \frac{2}{3}\sqrt{3}$$

16. Let $x = \frac{2}{3}\sin\theta$, so $dx = \frac{2}{3}\cos\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = \frac{2}{3} \;\Rightarrow\; \theta = \frac{\pi}{2}$. Thus,



$$\int_0^{2/3} \sqrt{4-9x^2}\,dx = \int_0^{\pi/2} \sqrt{4 - 9\cdot\frac{4}{9}\sin^2\theta}\;\frac{2}{3}\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 2\cos\theta\cdot\frac{2}{3}\cos\theta\,d\theta = \frac{4}{3}\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2}\frac{1}{2}(1+\cos 2\theta)\,d\theta = \frac{2}{3}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{2}{3}\left[\left(\frac{\pi}{2}+0\right) - (0+0)\right] = \frac{\pi}{3}$$

17. $$\int_0^{1/2} x\sqrt{1-4x^2}\,dx = \int_1^0 u^{1/2}\left(-\frac{1}{8}\,du\right) \qquad \left[\begin{matrix} u = 1-4x^2, \\ du = -8x\,dx \end{matrix}\right]$$

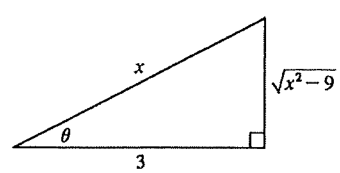
$$= \tfrac{1}{8}\left[\tfrac{2}{3}u^{3/2}\right]_0^1 = \tfrac{1}{12}(1-0) = \tfrac{1}{12}$$

18. Let $t = 2\tan\theta$, so $dt = 2\sec^2\theta\,d\theta$, $t = 0 \;\Rightarrow\; \theta = 0$, and $t = 2 \;\Rightarrow\; \theta = \frac{\pi}{4}$. Thus,

$$\int_0^2 \frac{dt}{\sqrt{4+t^2}} = \int_0^{\pi/4} \frac{2\sec^2\theta\,d\theta}{\sqrt{4+4\tan^2\theta}} = \int_0^{\pi/4} \frac{2\sec^2\theta\,d\theta}{2\sec\theta} = \int_0^{\pi/4}\sec\theta\,d\theta = \Big[\ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4}$$

$$= \ln\left|\sqrt{2}+1\right| - \ln|1+0| = \ln(\sqrt{2}+1)$$

19. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = 3\sec\theta\,\tan\theta\,d\theta$ and $\sqrt{x^2-9} = 3\tan\theta$, so



$$\int \frac{\sqrt{x^2-9}}{x^3}\,dx = \int \frac{3\tan\theta}{27\sec^3\theta}\,3\sec\theta\,\tan\theta\,d\theta = \frac{1}{3}\int\frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$= \tfrac{1}{3}\int\sin^2\theta\,d\theta = \tfrac{1}{3}\int\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{6}\theta - \tfrac{1}{12}\sin 2\theta + C = \tfrac{1}{6}\theta - \tfrac{1}{6}\sin\theta\,\cos\theta + C$$

$$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{6}\frac{\sqrt{x^2-9}}{x}\,\frac{3}{x} + C = \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C$$

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20. Let $x = \tan\theta$, so $dx = \sec^2\theta\, d\theta$, $x = 0 \Rightarrow \theta = 0$, and $x = 1 \Rightarrow \theta = \frac{\pi}{4}$. Then

$$\begin{aligned}\int_0^1 \frac{dx}{(x^2+1)^2} &= \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{(\tan^2\theta+1)^2} = \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^2} \\ &= \int_0^{\pi/4} \cos^2\theta\, d\theta = \int_0^{\pi/4} \frac{1}{2}(1+\cos 2\theta)\, d\theta \\ &= \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \tfrac{1}{2}\left[\left(\tfrac{\pi}{4}+\tfrac{1}{2}\right) - 0\right] = \tfrac{\pi}{8} + \tfrac{1}{4}\end{aligned}$$

21. Let $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$, $x = 0 \Rightarrow \theta = 0$ and $x = a \Rightarrow \theta = \frac{\pi}{2}$. Then

$$\begin{aligned}\int_0^a x^2\sqrt{a^2-x^2}\, dx &= \int_0^{\pi/2} a^2\sin^2\theta\,(a\cos\theta)\, a\cos\theta\, d\theta = a^4\int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta \\ &= a^4\int_0^{\pi/2}\left[\tfrac{1}{2}(2\sin\theta\cos\theta)\right]^2 d\theta = \frac{a^4}{4}\int_0^{\pi/2}\sin^2 2\theta\, d\theta = \frac{a^4}{4}\int_0^{\pi/2}\tfrac{1}{2}(1-\cos 4\theta)\, d\theta \\ &= \frac{a^4}{8}\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{a^4}{8}\left[\left(\frac{\pi}{2}-0\right)-0\right] = \frac{\pi}{16}a^4\end{aligned}$$

22. Let $x = \dfrac{1}{2}\sin\theta$, so $dx = \dfrac{1}{2}\cos\theta\, d\theta$, $x = \dfrac{1}{4} \Rightarrow \theta = \dfrac{\pi}{6}$, and $x = \dfrac{\sqrt{3}}{4} \Rightarrow \theta = \dfrac{\pi}{3}$. Then

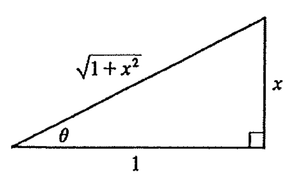
$$\begin{aligned}\int_{1/4}^{\sqrt{3}/4}\sqrt{1-4x^2}\, dx &= \int_{\pi/6}^{\pi/3}\sqrt{1-\sin^2\theta}\left(\frac{1}{2}\cos\theta\, d\theta\right) = \frac{1}{2}\int_{\pi/6}^{\pi/3}\cos^2\theta\, d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/3}\frac{1}{2}(1+\cos 2\theta)\, d\theta \\ &= \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/3} = \frac{1}{4}\left[\left(\frac{\pi}{3}+\frac{1}{2}\sin\frac{2\pi}{3}\right) - \left(\frac{\pi}{6}+\frac{1}{2}\sin\frac{\pi}{3}\right)\right] \\ &= \frac{1}{4}\left[\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right) - \left(\frac{\pi}{6}+\frac{\sqrt{3}}{4}\right)\right] = \frac{\pi}{24}\end{aligned}$$

23. Let $u = x^2 - 7$, so $du = 2x\, dx$. Then $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\, dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\, du = \tfrac{1}{2}\cdot 2\sqrt{u} + C = \sqrt{x^2-7} + C$.

24. Let $u = 1 + x^2$, so $du = 2x\, dx$. Then

$$\int \frac{x}{\sqrt{1+x^2}}\, dx = \int \frac{1}{\sqrt{u}}\left(\frac{1}{2}\, du\right) = \frac{1}{2}\int u^{-1/2}\, du = \frac{1}{2}\cdot 2u^{1/2} + C = \sqrt{1+x^2} + C$$

25. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\, d\theta$ and $\sqrt{1+x^2} = \sec\theta$, so



$$\begin{aligned}\int \frac{\sqrt{1+x^2}}{x}\, dx &= \int \frac{\sec\theta}{\tan\theta}\sec^2\theta\, d\theta = \int \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\, d\theta \\ &= \int (\csc\theta + \sec\theta\tan\theta)\, d\theta \\ &= \ln|\csc\theta - \cot\theta| + \sec\theta + C \qquad \text{[by Exercise 7.2.41]} \\ &= \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \frac{\sqrt{1+x^2}}{1} + C = \ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + \sqrt{1+x^2} + C\end{aligned}$$

Sec 7.3

26. Let $x = \frac{3}{5}\sin\theta$, so $dx = \frac{3}{5}\cos\theta\, d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 0.3 \;\Rightarrow\; \theta = \frac{\pi}{6}$. Then

$$\int_0^{0.3} \frac{x}{(9-25x^2)^{3/2}}\,dx = \int_0^{\pi/6} \frac{\frac{3}{5}\sin\theta}{\left(\sqrt{9-9\sin^2\theta}\right)^3}\left(\frac{3}{5}\cos\theta\,d\theta\right)$$

$$= \frac{9}{25}\int_0^{\pi/6} \frac{\sin\theta}{(3\cos\theta)^3}\cos\theta\,d\theta = \frac{1}{75}\int_0^{\pi/6}\frac{\sin\theta}{\cos^2\theta}\,d\theta \overset{c}{=} -\frac{1}{75}\int_1^{\sqrt{3}/2}\frac{1}{u^2}\,du$$

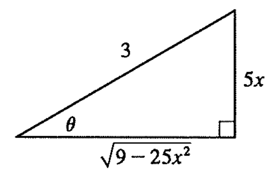
$$= -\frac{1}{75}\left[-\frac{1}{u}\right]_1^{\sqrt{3}/2} = \frac{1}{75}\left(\frac{2}{\sqrt{3}} - 1\right) = \frac{2\sqrt{3}}{225} - \frac{1}{75}$$

27. Let $x = \frac{3}{5}\sin\theta$, so $dx = \frac{3}{5}\cos\theta\, d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 0.6 \;\Rightarrow\; \theta = \frac{\pi}{2}$. Then

$$\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\,dx = \int_0^{\pi/2}\frac{\left(\frac{3}{5}\right)^2\sin^2\theta}{3\cos\theta}\left(\frac{3}{5}\cos\theta\,d\theta\right) = \frac{9}{125}\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= \frac{9}{125}\int_0^{\pi/2}\frac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{9}{250}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= \frac{9}{250}\left[\left(\frac{\pi}{2} - 0\right) - 0\right] = \frac{9}{500}\pi$$



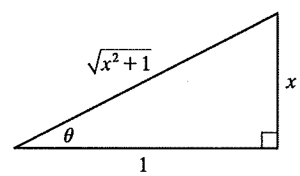
28. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$,

$\sqrt{x^2+1} = \sec\theta$ and $x = 0 \;\Rightarrow\; \theta = 0$, $x = 1 \;\Rightarrow\; \theta = \frac{\pi}{4}$, so

$$\int_0^1 \sqrt{x^2+1}\,dx = \int_0^{\pi/4}\sec\theta\,\sec^2\theta\,d\theta = \int_0^{\pi/4}\sec^3\theta\,d\theta$$

$$= \frac{1}{2}\Big[\sec\theta\,\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4} \quad \text{[by Example 7.2.8]}$$

$$= \frac{1}{2}\left[\sqrt{2}\cdot 1 + \ln\left(1+\sqrt{2}\right) - 0 - \ln(1+0)\right] = \frac{1}{2}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$$

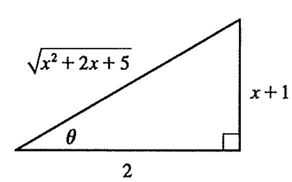


29. $$\int\frac{dx}{\sqrt{x^2+2x+5}} = \int\frac{dx}{\sqrt{(x+1)^2+4}} = \int\frac{2\sec^2\theta\,d\theta}{\sqrt{4\tan^2\theta+4}} \qquad \left[\begin{array}{r} x+1 = 2\tan\theta, \\ dx = 2\sec^2\theta\,d\theta \end{array}\right]$$

$$= \int\frac{2\sec^2\theta\,d\theta}{2\sec\theta} = \int\sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C_1$$

$$= \ln\left|\frac{\sqrt{x^2+2x+5}}{2} + \frac{x+1}{2}\right| + C_1,$$

or $\ln\left|\sqrt{x^2+2x+5} + x + 1\right| + C$, where $C = C_1 - \ln 2$.



30. $$\int_0^1\sqrt{x-x^2}\,dx = \int_0^1\sqrt{\tfrac{1}{4} - \left(x^2 - x + \tfrac{1}{4}\right)}\,dx = \int_0^1\sqrt{\tfrac{1}{4} - \left(x - \tfrac{1}{2}\right)^2}\,dx$$

$$= \int_{-\pi/2}^{\pi/2}\sqrt{\tfrac{1}{4} - \tfrac{1}{4}\sin^2\theta}\;\tfrac{1}{2}\cos\theta\,d\theta \qquad \left[\begin{array}{r} x - \frac{1}{2} = \frac{1}{2}\sin\theta, \\ dx = \frac{1}{2}\cos\theta\,d\theta \end{array}\right]$$

$$= 2\int_0^{\pi/2}\tfrac{1}{2}\cos\theta\,\tfrac{1}{2}\cos\theta\,d\theta = \tfrac{1}{2}\int_0^{\pi/2}\cos^2\theta\,d\theta = \tfrac{1}{2}\int_0^{\pi/2}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{4}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{1}{4}\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{8}$$