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## 7.4 Integration of Rational Functions by Partial Fractions

1. (a) $\dfrac{1}{(x-3)(x+5)} = \dfrac{A}{x-3} + \dfrac{B}{x+5}$

(b) $\dfrac{2x+5}{(x-2)^2(x^2+2)} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{Cx+D}{x^2+2}$

2. (a) $\dfrac{x-6}{x^2+x-6} = \dfrac{x-6}{(x+3)(x-2)} = \dfrac{A}{x+3} + \dfrac{B}{x-2}$

(b) $\dfrac{1}{x^2+x^4} = \dfrac{1}{x^2(1+x^2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{1+x^2}$

3. (a) $\dfrac{x^2+4}{x^3-3x^2+2x} = \dfrac{x^2+4}{x(x^2-3x+2)} = \dfrac{x^2+4}{x(x-1)(x-2)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x-2}$

(b) $\dfrac{x^3+x}{x(2x-1)^2(x^2+3)^2} = \dfrac{A}{x} + \dfrac{B}{2x-1} + \dfrac{C}{(2x-1)^2} + \dfrac{Dx+E}{x^2+3} + \dfrac{Fx+G}{(x^2+3)^2}$

4. (a) $\dfrac{5}{x^4-1} = \dfrac{5}{(x^2+1)(x^2-1)} = \dfrac{5}{(x^2+1)(x+1)(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{Cx+D}{x^2+1}$

(b) $\dfrac{x^4+x+1}{(x^3-1)(x^2-1)} = \dfrac{x^4+x+1}{(x-1)(x^2+x+1)(x+1)(x-1)} = \dfrac{x^4+x+1}{(x+1)(x-1)^2(x^2+x+1)}$

$$= \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{Dx+E}{x^2+x+1}$$

5. (a) $\dfrac{x^5+1}{(x^2-x)(x^4+2x^2+1)} = \dfrac{x^5+1}{x(x-1)(x^2+1)^2} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2}$

(b) $\dfrac{x^2}{x^2+x-6} = 1 + \dfrac{-x+6}{x^2+x-6} = 1 + \dfrac{-x+6}{(x-2)(x+3)} = 1 + \dfrac{A}{x-2} + \dfrac{B}{x+3}$

6. (a) $\dfrac{x^6}{x^2-4} = x^4 + 4x^2 + 16 + \dfrac{64}{(x+2)(x-2)}$ [by long division]

$$= x^4 + 4x^2 + 16 + \frac{A}{x+2} + \frac{B}{x-2}$$

(b) $\dfrac{x^4}{(x^2-x+1)(x^2+2)^2} = \dfrac{Ax+B}{x^2-x+1} + \dfrac{Cx+D}{x^2+2} + \dfrac{Ex+F}{(x^2+2)^2}$

7. $\dfrac{5}{(x-1)(x+4)} = \dfrac{A}{x-1} + \dfrac{B}{x+4}$. Multiply both sides by $(x-1)(x+4)$ to get $5 = A(x+4) + B(x-1) \Rightarrow$ $5 = (A+B)x + (4A-B)$. The coefficients of $x$ must be equal and the constant terms are also equal, so $A+B=0$ and $4A-B=5$. Adding the equations together gives $5A=5 \Leftrightarrow A=1$, and hence $B=-1$. Thus,

$$\int \frac{5}{(x-1)(x+4)}\,dx = \int \left(\frac{1}{x-1} - \frac{1}{x+4}\right) dx = \ln|x-1| - \ln|x+4| + C.$$

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**8.** $\dfrac{x-12}{x^2-4x} = \dfrac{x-12}{x(x-4)} = \dfrac{A}{x} + \dfrac{B}{x-4}$. Multiply both sides by $x(x-4)$ to get $x - 12 = A(x-4) + Bx \Rightarrow$

$x - 12 = (A+B)x + (-4A)$. The coefficients of $x$ must be equal and the constant terms are also equal, so $A + B = 1$ and $-4A = -12$. The second equation gives $A = 3$, which after substituting in the first equation gives $B = -2$. Thus,

$$\int \frac{x-12}{x^2-4x}\,dx = \int \left(\frac{3}{x} - \frac{2}{x-4}\right) dx = 3\ln|x| - 2\ln|x-4| + C.$$

**9.** $\dfrac{5x+1}{(2x+1)(x-1)} = \dfrac{A}{2x+1} + \dfrac{B}{x-1}$. Multiply both sides by $(2x+1)(x-1)$ to get $5x+1 = A(x-1) + B(2x+1) \Rightarrow$

$5x + 1 = Ax - A + 2Bx + B \quad\Rightarrow\quad 5x + 1 = (A+2B)x + (-A+B)$.

The coefficients of $x$ must be equal and the constant terms are also equal, so $A + 2B = 5$ and $-A + B = 1$. Adding these equations gives us $3B = 6 \quad\Leftrightarrow\quad B = 2$, and hence, $A = 1$. Thus,

$$\int \frac{5x+1}{(2x+1)(x-1)}\,dx = \int \left(\frac{1}{2x+1} + \frac{2}{x-1}\right) dx = \tfrac{1}{2}\ln|2x+1| + 2\ln|x-1| + C.$$

*Another method:* Substituting 1 for $x$ in the equation $5x + 1 = A(x-1) + B(2x+1)$ gives $6 = 3B \quad\Leftrightarrow\quad B = 2$. Substituting $-\frac{1}{2}$ for $x$ gives $-\frac{3}{2} = -\frac{3}{2}A \quad\Leftrightarrow\quad A = 1$.

**10.** $\dfrac{y}{(y+4)(2y-1)} = \dfrac{A}{y+4} + \dfrac{B}{2y-1}$. Multiply both sides by $(y+4)(2y-1)$ to get $y = A(2y-1) + B(y+4) \Rightarrow$

$y = 2Ay - A + By + 4B \quad\Rightarrow\quad y = (2A+B)y + (-A+4B)$. The coefficients of $y$ must be equal and the constant terms are also equal, so $2A + B = 1$ and $-A + 4B = 0$. Adding 2 times the second equation and the first equation gives us $9B = 1 \quad\Leftrightarrow\quad B = \frac{1}{9}$ and hence, $A = \frac{4}{9}$. Thus,

$$\begin{aligned}\int \frac{y}{(y+4)(2y-1)}\,dy &= \int \left(\frac{\frac{4}{9}}{y+4} + \frac{\frac{1}{9}}{2y-1}\right) dy = \frac{4}{9}\ln|y+4| + \frac{1}{9}\cdot\frac{1}{2}\ln|2y-1| + C \\ &= \tfrac{4}{9}\ln|y+4| + \tfrac{1}{18}\ln|2y-1| + C\end{aligned}$$

*Another method:* Substituting $\frac{1}{2}$ for $y$ in the equation $y = A(2y-1) + B(y+4)$ gives $\frac{1}{2} = \frac{9}{2}B \quad\Leftrightarrow\quad B = \frac{1}{9}$. Substituting $-4$ for $y$ gives $-4 = -9A \quad\Leftrightarrow\quad A = \frac{4}{9}$.

**11.** $\dfrac{2}{2x^2+3x+1} = \dfrac{2}{(2x+1)(x+1)} = \dfrac{A}{2x+1} + \dfrac{B}{x+1}$. Multiply both sides by $(2x+1)(x+1)$ to get

$2 = A(x+1) + B(2x+1)$. The coefficients of $x$ must be equal and the constant terms are also equal, so $A + 2B = 0$ and $A + B = 2$. Subtracting the second equation from the first gives $B = -2$, and hence, $A = 4$. Thus,

$$\int_0^1 \frac{2}{2x^2+3x+1}\,dx = \int_0^1 \left(\frac{4}{2x+1} - \frac{2}{x+1}\right) dx = \left[\frac{4}{2}\ln|2x+1| - 2\ln|x+1|\right]_0^1 = (2\ln 3 - 2\ln 2) - 0 = 2\ln\frac{3}{2}.$$

*Another method:* Substituting $-1$ for $x$ in the equation $2 = A(x+1) + B(2x+1)$ gives $2 = -B \quad\Leftrightarrow\quad B = -2$. Substituting $-\frac{1}{2}$ for $x$ gives $2 = \frac{1}{2}A \quad\Leftrightarrow\quad A = 4$.

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**12.** $\dfrac{x-4}{x^2-5x+6} = \dfrac{A}{x-2} + \dfrac{B}{x-3}$. Multiply both sides by $(x-2)(x-3)$ to get $x-4 = A(x-3) + B(x-2) \Rightarrow$

$x - 4 = Ax - 3A + Bx - 2B \Rightarrow x - 4 = (A+B)x + (-3A - 2B)$.

The coefficients of $x$ must be equal and the constant terms are also equal, so $A + B = 1$ and $-3A - 2B = -4$.

Adding twice the first equation to the second gives us $-A = -2 \Leftrightarrow A = 2$, and hence, $B = -1$.Thus,

$$\begin{aligned}\int_0^1 \frac{x-4}{x^2-5x+6}\,dx &= \int_0^1 \left(\frac{2}{x-2} - \frac{1}{x-3}\right) dx = \big[2\ln|x-2| - \ln|x-3|\big]_0^1 \\ &= (0 - \ln 2) - (2\ln 2 - \ln 3) = -3\ln 2 + \ln 3 \;\; [\text{or } \ln \tfrac{3}{8}]\end{aligned}$$

*Another method:* Substituting 3 for $x$ in the equation $x - 4 = A(x-3) + B(x-2)$ gives $-1 = B$. Substituting 2 for $x$ gives $-2 = -A \Leftrightarrow A = 2$.

**13.** $\dfrac{1}{x(x-a)} = \dfrac{A}{x} + \dfrac{B}{x-a}$. Multiply both sides by $x(x-a)$ to get $1 = A(x-a) + Bx \Rightarrow 1 = (A+B)x + (-aA)$.

The coefficients of $x$ must be equal and the constant terms are also equal, so $A + B = 0$ and $-aA = 1$. The second equation gives $A = -1/a$, which after substituting in the first equation gives $B = 1/a$. Thus,

$$\int \frac{1}{x(x-a)}\,dx = \int \left(-\frac{1/a}{x} + \frac{1/a}{x-a}\right) dx = -\frac{1}{a}\ln|x| + \frac{1}{a}\ln|x-a| + C.$$

**14.** If $a \neq b$, $\dfrac{1}{(x+a)(x+b)} = \dfrac{1}{b-a}\left(\dfrac{1}{x+a} - \dfrac{1}{x+b}\right)$, so if $a \neq b$, then

$$\int \frac{1}{(x+a)(x+b)}\,dx = \frac{1}{b-a}\left(\ln|x+a| - \ln|x+b|\right) + C = \frac{1}{b-a}\ln\left|\frac{x+a}{x+b}\right| + C$$

If $a = b$, then $\displaystyle\int \frac{1}{(x+a)^2}\,dx = -\frac{1}{x+a} + C$.

**15.** $\dfrac{x^2}{x-1} = \dfrac{(x^2-1)+1}{x-1} = \dfrac{(x+1)(x-1)+1}{x-1} = x + 1 + \dfrac{1}{x-1}$. [This result can also be obtained using long division.]

Thus, $\displaystyle\int \frac{x^2}{x-1}\,dx = \int \left(x + 1 + \frac{1}{x-1}\right) dx = \frac{1}{2}x^2 + x + \ln|x-1| + C$.

**16.** $\dfrac{3t-2}{t+1} = \dfrac{3t+3-5}{t+1} = \dfrac{3(t+1)-5}{t+1} = 3 - \dfrac{5}{t+1}$. Thus, $\displaystyle\int \frac{3t-2}{t+1}\,dt = \int \left(3 - \frac{5}{t+1}\right) dt = 3t - 5\ln|t+1| + C$.

**17.** $\dfrac{4y^2-7y-12}{y(y+2)(y-3)} = \dfrac{A}{y} + \dfrac{B}{y+2} + \dfrac{C}{y-3} \Rightarrow 4y^2 - 7y - 12 = A(y+2)(y-3) + By(y-3) + Cy(y+2)$. Setting $y = 0$ gives $-12 = -6A$, so $A = 2$. Setting $y = -2$ gives $18 = 10B$, so $B = \frac{9}{5}$. Setting $y = 3$ gives $3 = 15C$, so $C = \frac{1}{5}$.

Now

$$\begin{aligned}\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy &= \int_1^2 \left(\frac{2}{y} + \frac{9/5}{y+2} + \frac{1/5}{y-3}\right) dy = \left[2\ln|y| + \tfrac{9}{5}\ln|y+2| + \tfrac{1}{5}\ln|y-3|\right]_1^2 \\ &= 2\ln 2 + \tfrac{9}{5}\ln 4 + \tfrac{1}{5}\ln 1 - 2\ln 1 - \tfrac{9}{5}\ln 3 - \tfrac{1}{5}\ln 2 \\ &= 2\ln 2 + \tfrac{18}{5}\ln 2 - \tfrac{1}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{27}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{9}{5}(3\ln 2 - \ln 3) = \tfrac{9}{5}\ln\tfrac{8}{3}\end{aligned}$$

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**18.** $\dfrac{3x^2+6x+2}{x^2+3x+2} = 3 + \dfrac{-3x-4}{(x+1)(x+2)}$. Write $\dfrac{-3x-4}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$. Multiplying both sides by $(x+1)(x+2)$ gives $-3x-4 = A(x+2) + B(x+1)$. Substituting $-2$ for $x$ gives $2 = -B \iff B = -2$. Substituting $-1$ for $x$ gives $-1 = A$. Thus,

$$\int_1^2 \frac{3x^2+6x+2}{x^2+3x+2}\,dx = \int_1^2 \left(3 - \frac{1}{x+1} - \frac{2}{x+2}\right) dx = \Big[3x - \ln|x+1| - 2\ln|x+2|\Big]_1^2$$
$$= (6 - \ln 3 - 2\ln 4) - (3 - \ln 2 - 2\ln 3) = 3 + \ln 2 + \ln 3 - 2\ln 4, \text{ or } 3 + \ln\tfrac{3}{8}$$

**19.** $\dfrac{x^2+x+1}{(x+1)^2(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+2}$. Multiplying both sides by $(x+1)^2(x+2)$ gives $x^2+x+1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$. Substituting $-1$ for $x$ gives $1 = B$. Substituting $-2$ for $x$ gives $3 = C$. Equating coefficients of $x^2$ gives $1 = A + C = A + 3$, so $A = -2$. Thus,

$$\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx = \int_0^1 \left(\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}\right) dx = \left[-2\ln|x+1| - \frac{1}{x+1} + 3\ln|x+2|\right]_0^1$$
$$= \left(-2\ln 2 - \tfrac{1}{2} + 3\ln 3\right) - (0 - 1 + 3\ln 2) = \tfrac{1}{2} - 5\ln 2 + 3\ln 3, \text{ or } \tfrac{1}{2} + \ln\tfrac{27}{32}$$

**20.** $\dfrac{x(3-5x)}{(3x-1)(x-1)^2} = \dfrac{A}{3x-1} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$. Multiplying both sides by $(3x-1)(x-1)^2$ gives $x(3-5x) = A(x-1)^2 + B(x-1)(3x-1) + C(3x-1)$. Substituting 1 for $x$ gives $-2 = 2C \iff C = -1$. Substituting $\frac{1}{3}$ for $x$ gives $\frac{4}{9} = \frac{4}{9}A \iff A = 1$. Substituting 0 for $x$ gives $0 = A + B - C = 1 + B + 1$, so $B = -2$. Thus,

$$\int_2^3 \frac{x(3-5x)}{(3x-1)(x-1)^2}\,dx = \int_2^3 \left[\frac{1}{3x-1} - \frac{2}{x-1} - \frac{1}{(x-1)^2}\right] dx = \left[\frac{1}{3}\ln|3x-1| - 2\ln|x-1| + \frac{1}{x-1}\right]_2^3$$
$$= \left(\tfrac{1}{3}\ln 8 - 2\ln 2 + \tfrac{1}{2}\right) - \left(\tfrac{1}{3}\ln 5 - 0 + 1\right) = -\ln 2 - \tfrac{1}{3}\ln 5 - \tfrac{1}{2}$$

**21.** $\dfrac{1}{(t^2-1)^2} = \dfrac{1}{(t+1)^2(t-1)^2} = \dfrac{A}{t+1} + \dfrac{B}{(t+1)^2} + \dfrac{C}{t-1} + \dfrac{D}{(t-1)^2}$. Multiplying both sides by $(t+1)^2(t-1)^2$ gives $1 = A(t+1)(t-1)^2 + B(t-1)^2 + C(t-1)(t+1)^2 + D(t+1)^2$. Substituting 1 for $t$ gives $1 = 4D \iff D = \frac{1}{4}$. Substituting $-1$ for $t$ gives $1 = 4B \iff B = \frac{1}{4}$. Substituting 0 for $t$ gives $1 = A + B - C + D = A + \frac{1}{4} - C + \frac{1}{4}$, so $\frac{1}{2} = A - C$. Equating coefficients of $t^3$ gives $0 = A + C$. Adding the last two equations gives $2A = \frac{1}{2} \iff A = \frac{1}{4}$, and so $C = -\frac{1}{4}$. Thus,

$$\int \frac{dt}{(t^2-1)^2} = \int \left[\frac{1/4}{t+1} + \frac{1/4}{(t+1)^2} - \frac{1/4}{t-1} + \frac{1/4}{(t-1)^2}\right] dt$$
$$= \frac{1}{4}\left[\ln|t+1| - \frac{1}{t+1} - \ln|t-1| - \frac{1}{t-1}\right] + C, \text{ or } \frac{1}{4}\left(\ln\left|\frac{t+1}{t-1}\right| + \frac{2t}{1-t^2}\right) + C$$

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**22.** $\dfrac{3x^2+12x-20}{x^4-8x^2+16}=\dfrac{3x^2+12x-20}{(x^2-4)^2}=\dfrac{3x^2+12x-20}{(x-2)^2(x+2)^2}=\dfrac{A}{x-2}+\dfrac{B}{(x-2)^2}+\dfrac{C}{x+2}+\dfrac{D}{(x+2)^2}$. Multiply both sides by $(x-2)^2(x+2)^2$ to get $3x^2+12x-20=A(x-2)(x+2)^2+B(x+2)^2+C(x-2)^2(x+2)+D(x-2)^2$. Setting $x=2$ gives $16=16B$, so $B=1$, and setting $x=-2$ gives $-32=16D$, so $D=-2$. Now, using these values of $B$ and $D$ and setting $x=0$ gives $-20=-8A+4+8C-8 \quad\Leftrightarrow\quad -2=-A+C$ **(1)**. Also, setting $x=1$ gives $-5=-9A+9+3C-2 \quad\Leftrightarrow\quad -4=-3A+C$ **(2)**. Subtracting **(2)** from **(1)** gives $2=2A \quad\Leftrightarrow\quad A=1$, and hence $C=-1$. Thus,

$$\begin{aligned}\int\frac{3x^2+12x-20}{x^4-8x^2+16}\,dx&=\int\left(\frac{1}{x-2}+\frac{1}{(x-2)^2}-\frac{1}{x+2}-\frac{2}{(x+2)^2}\right)dx\\&=\ln|x-2|-\frac{1}{x-2}-\ln|x+2|+\frac{2}{x+2}+C\end{aligned}$$

**23.** $\dfrac{10}{(x-1)(x^2+9)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+9}$. Multiply both sides by $(x-1)(x^2+9)$ to get $10=A(x^2+9)+(Bx+C)(x-1)$ $(\star)$. Substituting 1 for $x$ gives $10=10A \quad\Leftrightarrow\quad A=1$. Substituting 0 for $x$ gives $10=9A-C \quad\Rightarrow\quad C=9(1)-10=-1$. The coefficients of the $x^2$-terms in $(\star)$ must be equal, so $0=A+B \quad\Rightarrow\quad B=-1$. Thus,

$$\begin{aligned}\int\frac{10}{(x-1)(x^2+9)}\,dx&=\int\left(\frac{1}{x-1}+\frac{-x-1}{x^2+9}\right)dx=\int\left(\frac{1}{x-1}-\frac{x}{x^2+9}-\frac{1}{x^2+9}\right)dx\\&=\ln|x-1|-\tfrac{1}{2}\ln(x^2+9)-\tfrac{1}{3}\tan^{-1}\left(\tfrac{x}{3}\right)+C\end{aligned}$$

In the second term we used the substitution $u=x^2+9$ and in the last term we used Formula 10.

**24.** $\dfrac{3x^2-x+8}{x^3+4x}=\dfrac{3x^2-x+8}{x(x^2+4)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+4}$. Multiply both sides by $x(x^2+4)$ to get $3x^2-x+8=A(x^2+4)+x(Bx+C) \quad\Rightarrow\quad 3x^2-x+8=(A+B)x^2+Cx+4A$. Equating constant terms, we get $4A=8 \quad\Leftrightarrow\quad A=2$. Equating coefficients of $x$ gives $C=-1$. Now equating coefficients of $x^2$ gives $A+B=3$, so $B=1$. Thus,

$$\begin{aligned}\int\frac{3x^2-x+8}{x^3+4x}\,dx&=\int\left(\frac{2}{x}+\frac{x-1}{x^2+4}\right)dx=\int\frac{2}{x}+\frac{x}{x^2+4}-\frac{1}{x^2+4}dx\\&=2\ln(x)+\tfrac{1}{2}\ln(x^2+4)-\tfrac{1}{2}\tan^{-1}\left(\tfrac{x}{2}\right)+C\end{aligned}$$

**25.** $\dfrac{x^3-4x+1}{x^2-3x+2}=x+3+\dfrac{3x-5}{(x-1)(x-2)}$. Write $\dfrac{3x-5}{(x-1)(x-2)}=\dfrac{A}{x-1}+\dfrac{B}{x-2}$. Multiplying both sides by $(x-1)(x-2)$ gives $3x-5=A(x-2)+B(x-1)$. Substituting 2 for $x$ gives $1=B$. Substituting 1 for $x$ gives $-2=-A \quad\Leftrightarrow\quad A=2$. Thus,

$$\begin{aligned}\int_{-1}^{0}\frac{x^3-4x+1}{x^2-3x+2}\,dx&=\int_{-1}^{0}\left(x+3+\frac{2}{x-1}+\frac{1}{x-2}\right)dx=\left[\tfrac{1}{2}x^2+3x+2\ln|x-1|+\ln|x-2|\right]_{-1}^{0}\\&=(0+0+0+\ln 2)-\left(\tfrac{1}{2}-3+2\ln 2+\ln 3\right)=\tfrac{5}{2}-\ln 2-\ln 3,\text{ or }\tfrac{5}{2}-\ln 6\end{aligned}$$