HW #3, Sec 11.2 Solutions

22. Using partial fractions, the partial sums of the series $\sum_{n=2}^{\infty} \frac{1}{n^3 - n}$ are

$$s_n = \sum_{i=2}^{n} \frac{1}{i(i-1)(i+1)} = \sum_{i=2}^{n} \left(-\frac{1}{i} + \frac{1/2}{i-1} + \frac{1/2}{i+1} \right) = \frac{1}{2} \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{2}{i} + \frac{1}{i+1} \right)$$

$$= \frac{1}{2} \left[\left(\frac{1}{1} - \frac{2}{2} + \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6} \right) + \cdots \right.$$

$$\left. + \left(\frac{1}{n-3} - \frac{2}{n-2} + \frac{1}{n-1} \right) + \left(\frac{1}{n-2} - \frac{2}{n-1} + \frac{1}{n} \right) + \left(\frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1} \right) \right]$$

Note: In three consecutive expressions in parentheses, the 3rd term in the first expression plus the 2nd term in the second expression plus the 1st term in the third expression sum to 0.

$$= \frac{1}{2} \left(\frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n} - \frac{2}{n} + \frac{1}{n+1} \right) = \frac{1}{4} - \frac{1}{2n} + \frac{1}{2n+2}$$

Thus, $\sum_{n=2}^{\infty} \frac{1}{n^3 - n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{4} - \frac{1}{2n} + \frac{1}{2n+2} \right) = \frac{1}{4}$.

23. $3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots$ is a geometric series with ratio $r = -\frac{4}{3}$. Since $|r| = \frac{4}{3} > 1$, the series diverges.

24. $4 + 3 + \frac{9}{4} + \frac{27}{16} + \cdots$ is a geometric series with ratio $\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, the series converges to $\frac{a}{1-r} = \frac{4}{1-3/4} = 16$.

25. $10 - 2 + 0.4 - 0.08 + \cdots$ is a geometric series with ratio $-\frac{2}{10} = -\frac{1}{5}$. Since $|r| = \frac{1}{5} < 1$, the series converges to

$$\frac{a}{1-r} = \frac{10}{1-(-1/5)} = \frac{10}{6/5} = \frac{50}{6} = \frac{25}{3}.$$

26. $2 + 0.5 + 0.125 + 0.03125 + \cdots$ is a geometric series with ratio $r = \frac{0.5}{2} = \frac{1}{4}$. Since $|r| = \frac{1}{4} < 1$, the series converges

to $\frac{a}{1-r} = \frac{2}{1-1/4} = \frac{2}{3/4} = \frac{8}{3}$.

27. $\sum_{n=1}^{\infty} 12\,(0.73)^{n-1}$ is a geometric series with first term $a = 12$ and ratio $r = 0.73$. Since $|r| = 0.73 < 1$, the series converges

to $\frac{a}{1-r} = \frac{12}{1-0.73} = \frac{12}{0.27} = \frac{12(100)}{27} = \frac{400}{9}$.

28. $\sum_{n=1}^{\infty} \frac{5}{\pi^n} = 5 \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \right)^n$. The latter series is geometric with $a = \frac{1}{\pi}$ and ratio $r = \frac{1}{\pi}$. Since $|r| = \frac{1}{\pi} < 1$, it converges to

$\frac{1/\pi}{1-1/\pi} = \frac{1}{\pi - 1}$. Thus, the given series converges to $5\left(\frac{1}{\pi-1}\right) = \frac{5}{\pi - 1}$.

29. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{4} \sum_{n=1}^{\infty} \left(-\frac{3}{4} \right)^{n-1}$. The latter series is geometric with $a = 1$ and ratio $r = -\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, it

converges to $\frac{1}{1-(-3/4)} = \frac{4}{7}$. Thus, the given series converges to $\left(\frac{1}{4}\right)\left(\frac{4}{7}\right) = \frac{1}{7}$.

30. $\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^n} = 3\sum_{n=0}^{\infty} \left(-\frac{3}{2}\right)^n$ is a geometric series with ratio $r = -\frac{3}{2}$. Since $|r| = \frac{3}{2} > 1$, the series diverges.

31. $\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{(e^2)^n}{6^n 6^{-1}} = 6\sum_{n=1}^{\infty} \left(\frac{e^2}{6}\right)^n$ is a geometric series with ratio $r = \frac{e^2}{6}$. Since $|r| = \frac{e^2}{6} [\approx 1.23] > 1$, the series diverges.

32. $\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n} = \sum_{n=1}^{\infty} \frac{6(2^2)^n \cdot 2^{-1}}{3^n} = 3\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$ is a geometric series with ratio $r = \frac{4}{3}$. Since $|r| = \frac{4}{3} > 1$, the series diverges.

33. $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n}$. This is a constant multiple of the divergent harmonic series, so it diverges.

34. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7} + \cdots = \sum_{n=1}^{\infty} \frac{n}{n+1}$. This series diverges by the Test for Divergence since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1 \neq 0$.

35. $\frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \frac{16}{625} + \frac{32}{3125} + \cdots = \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$. This series is geometric with $a = \frac{2}{5}$ and ratio $r = \frac{2}{5}$. Since $|r| = \frac{2}{5} < 1$, it converges to $\frac{2/5}{1-2/5} = \frac{2}{3}$.

36. $\frac{1}{3} + \frac{2}{9} + \frac{1}{27} + \frac{2}{81} + \frac{1}{243} + \frac{2}{729} + \cdots = \left(\frac{1}{3} + \frac{1}{27} + \frac{1}{243} + \cdots\right) + \left(\frac{2}{9} + \frac{2}{81} + \frac{2}{729} + \cdots\right)$, which are both convergent geometric series with sums $\frac{1/3}{1-1/9} = \frac{3}{8}$ and $\frac{2/9}{1-1/9} = \frac{1}{4}$, so the original series converges and its sum is $\frac{3}{8} + \frac{1}{4} = \frac{5}{8}$.

37. $\sum_{n=1}^{\infty} \frac{2+n}{1-2n}$ diverges by the Test for Divergence since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2+n}{1-2n} = \lim_{n\to\infty} \frac{2/n+1}{1/n-2} = -\frac{1}{2} \neq 0$.

38. $\sum_{k=1}^{\infty} \frac{k^2}{k^2-2k+5}$ diverges by the Test for Divergence since $\lim_{k\to\infty} \frac{k^2}{k^2-2k+5} = \lim_{k\to\infty} \frac{1}{1-2/k+5/k^2} = 1 \neq 0$.

39. $\sum_{n=1}^{\infty} 3^{n+1}4^{-n} = \sum_{n=1}^{\infty} \frac{3^n \cdot 3^1}{4^n} = 3\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$. The latter series is geometric with $a = \frac{3}{4}$ and ratio $r = \frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, it converges to $\frac{3/4}{1-3/4} = 3$. Thus, the given series converges to $3(3) = 9$.

40. $\sum_{n=1}^{\infty} [(-0.2)^n + (0.6)^{n-1}] = \sum_{n=1}^{\infty} (-0.2)^n + \sum_{n=1}^{\infty} (0.6)^{n-1}$ [sum of two geometric series]

$$= \frac{-0.2}{1-(-0.2)} + \frac{1}{1-0.6} = -\frac{1}{6} + \frac{5}{2} = \frac{7}{3}$$

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41. $\sum_{n=1}^{\infty} \frac{1}{4+e^{-n}}$ diverges by the Test for Divergence since $\lim_{n\to\infty} \frac{1}{4+e^{-n}} = \frac{1}{4+0} = \frac{1}{4} \neq 0$.

42. $\sum_{n=1}^{\infty} \frac{2^n+4^n}{e^n}$ diverges by the Test for Divergence since $\lim_{n\to\infty} \frac{2^n+4^n}{e^n} = \lim_{n\to\infty}\left(\frac{2^n}{e^n}+\frac{4^n}{e^n}\right) \geq \lim_{n\to\infty}\left(\frac{4}{e}\right)^n = \infty$

since $\frac{4}{e} > 1$.

43. $\sum_{k=1}^{\infty} (\sin 100)^k$ is a geometric series with first term $a = \sin 100$ [≈ -0.506] and ratio $r = \sin 100$. Since $|r| < 1$, the series

converges to $\frac{\sin 100}{1-\sin 100} \approx -0.336$.

44. $\sum_{n=1}^{\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n}$ diverges by the Test for Divergence since $\lim_{n\to\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n} = \frac{1}{1+0} = 1 \neq 0$.

45. $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$ diverges by the Test for Divergence since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right) = \ln\left(\lim_{n\to\infty}\frac{n^2+1}{2n^2+1}\right) = \ln\tfrac{1}{2} \neq 0.$$

46. $\sum_{k=0}^{\infty} (\sqrt{2})^{-k} = \sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^k$ is a geometric series with first term $a = \left(\frac{1}{\sqrt{2}}\right)^0 = 1$ and ratio $r = \frac{1}{\sqrt{2}}$. Since $|r| < 1$, the

series converges to $\frac{1}{1-1/\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}-1} \approx 3.414$.

47. $\sum_{n=1}^{\infty} \arctan n$ diverges by the Test for Divergence since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$.

48. $\sum_{n=1}^{\infty}\left(\frac{3}{5^n}+\frac{2}{n}\right)$ diverges because $\sum_{n=1}^{\infty}\frac{2}{n} = 2\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. (If it converged, then $\frac{1}{2}\cdot 2\sum_{n=1}^{\infty}\frac{1}{n}$ would also converge by Theorem 8(i), but we know from Example 9 that the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.) If the given series converges, then the difference $\sum_{n=1}^{\infty}\left(\frac{3}{5^n}+\frac{2}{n}\right) - \sum_{n=1}^{\infty}\frac{3}{5^n}$ must converge (since $\sum_{n=1}^{\infty}\frac{3}{5^n}$ is a convergent geometric series) and equal $\sum_{n=1}^{\infty}\frac{2}{n}$, but we have just seen that $\sum_{n=1}^{\infty}\frac{2}{n}$ diverges, so the given series must also diverge.

49. $\sum_{n=1}^{\infty}\frac{1}{e^n} = \sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^n$ is a geometric series with first term $a = \frac{1}{e}$ and ratio $r = \frac{1}{e}$. Since $|r| = \frac{1}{e} < 1$, the series converges to $\frac{1/e}{1-1/e} = \frac{1/e}{1-1/e}\cdot\frac{e}{e} = \frac{1}{e-1}$. By Example 8, $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$. Thus, by Theorem 8(ii),

$$\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right) = \sum_{n=1}^{\infty}\frac{1}{e^n} + \sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \frac{1}{e-1}+1 = \frac{1}{e-1}+\frac{e-1}{e-1} = \frac{e}{e-1}.$$

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59. $\sum_{n=1}^{\infty}(-5)^n x^n = \sum_{n=1}^{\infty}(-5x)^n$ is a geometric series with $r = -5x$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$

$|-5x| < 1 \Leftrightarrow |x| < \frac{1}{5}$, that is, $-\frac{1}{5} < x < \frac{1}{5}$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{-5x}{1-(-5x)} = \dfrac{-5x}{1+5x}$.

60. $\sum_{n=1}^{\infty}(x+2)^n$ is a geometric series with $r = x+2$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $|x+2| < 1$ $\Leftrightarrow$

$-1 < x+2 < 1 \Leftrightarrow -3 < x < -1$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{x+2}{1-(x+2)} = \dfrac{x+2}{-x-1}$.

61. $\sum_{n=0}^{\infty}\dfrac{(x-2)^n}{3^n} = \sum_{n=0}^{\infty}\left(\dfrac{x-2}{3}\right)^n$ is a geometric series with $r = \dfrac{x-2}{3}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$

$\left|\dfrac{x-2}{3}\right| < 1 \Leftrightarrow -1 < \dfrac{x-2}{3} < 1 \Leftrightarrow -3 < x-2 < 3 \Leftrightarrow -1 < x < 5$. In that case, the sum of the series is

$$\frac{a}{1-r} = \frac{1}{1-\frac{x-2}{3}} = \frac{1}{\frac{3-(x-2)}{3}} = \frac{3}{5-x}.$$

62. $\sum_{n=0}^{\infty}(-4)^n(x-5)^n = \sum_{n=0}^{\infty}[-4(x-5)]^n$ is a geometric series with $r = -4(x-5)$, so the series converges $\Leftrightarrow$

$|r| < 1 \Leftrightarrow |-4(x-5)| < 1 \Leftrightarrow |x-5| < \frac{1}{4} \Leftrightarrow -\frac{1}{4} < x-5 < \frac{1}{4} \Leftrightarrow \frac{19}{4} < x < \frac{21}{4}$. In that case, the sum of

the series is $\dfrac{a}{1-r} = \dfrac{1}{1-[-4(x-5)]} = \dfrac{1}{4x-19}$.

63. $\sum_{n=0}^{\infty}\dfrac{2^n}{x^n} = \sum_{n=0}^{\infty}\left(\dfrac{2}{x}\right)^n$ is a geometric series with $r = \dfrac{2}{x}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\left|\dfrac{2}{x}\right| < 1$ $\Leftrightarrow$

$2 < |x| \Leftrightarrow x > 2$ or $x < -2$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{1}{1-2/x} = \dfrac{x}{x-2}$.

64. $\sum_{n=0}^{\infty}\dfrac{x^n}{2^n} = \sum_{n=0}^{\infty}\left(\dfrac{x}{2}\right)^n$ is a geometric series with $r = \dfrac{x}{2}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\left|\dfrac{x}{2}\right| < 1$ $\Leftrightarrow$

$-1 < \dfrac{x}{2} < 1 \Leftrightarrow -2 < x < 2$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{1}{1-\frac{x}{2}} = \dfrac{2}{2-x}$.

65. $\sum_{n=0}^{\infty}e^{nx} = \sum_{n=0}^{\infty}(e^x)^n$ is a geometric series with $r = e^x$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $|e^x| < 1$ $\Leftrightarrow$

$-1 < e^x < 1 \Leftrightarrow 0 < e^x < 1 \Leftrightarrow x < 0$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{1}{1-e^x}$.

66. $\sum_{n=0}^{\infty}\dfrac{\sin^n x}{3^n} = \sum_{n=0}^{\infty}\left(\dfrac{\sin x}{3}\right)^n$ is a geometric series with $r = \dfrac{\sin x}{3}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$

$\left|\dfrac{\sin x}{3}\right| < 1 \Leftrightarrow |\sin x| < 3$, which is true for all x. Thus, the sum of the series is $\dfrac{a}{1-r} = \dfrac{1}{1-(\sin x)/3} = \dfrac{3}{3-\sin x}$.

67. After defining f, We use `convert(f,parfrac);` in Maple or `Apart` in Mathematica to find that the general term is

$\dfrac{3n^2+3n+1}{(n^2+n)^3} = \dfrac{1}{n^3} - \dfrac{1}{(n+1)^3}$. So the nth partial sum is

$$s_n = \sum_{k=1}^{n}\left(\frac{1}{k^3} - \frac{1}{(k+1)^3}\right) = \left(1 - \frac{1}{2^3}\right) + \left(\frac{1}{2^3} - \frac{1}{3^3}\right) + \cdots + \left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) = 1 - \frac{1}{(n+1)^3}$$