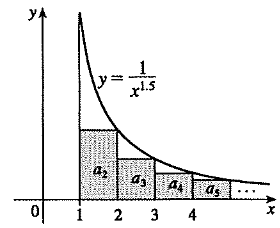
HW #3, Sec 11.3 Solutions

11.3 The Integral Test and Estimates of Sums

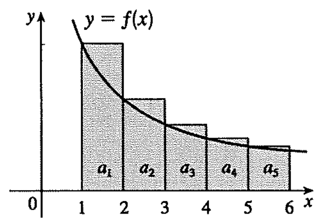
1. The picture shows that $a_2 = \dfrac{1}{2^{1.5}} < \displaystyle\int_1^2 \frac{1}{x^{1.5}}\,dx$,

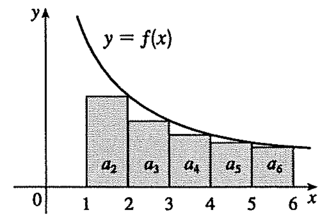
$a_3 = \dfrac{1}{3^{1.5}} < \displaystyle\int_2^3 \frac{1}{x^{1.5}}\,dx$, and so on, so $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1.5}} < \int_1^{\infty} \frac{1}{x^{1.5}}\,dx$.

The integral converges by (7.8.2) with $p = 1.5 > 1$, so the series converges.



2. From the first figure, we see that $\int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$. From the second figure, we see that $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx$. Thus, we have $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$.





3. The function $f(x) = x^{-3}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} x^{-3}\,dx = \lim_{t\to\infty} \int_1^t x^{-3}\,dx = \lim_{t\to\infty} \left[\frac{x^{-2}}{-2}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2} + \frac{1}{2}\right) = \frac{1}{2}.$$

Since this improper integral is convergent, the series $\displaystyle\sum_{n=1}^{\infty} n^{-3}$ is also convergent by the Integral Test.

4. The function $f(x) = x^{-0.3}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} x^{-0.3}\,dx = \lim_{t\to\infty} \int_1^t x^{-0.3}\,dx = \lim_{t\to\infty} \left[\frac{x^{0.7}}{0.7}\right]_1^t = \lim_{t\to\infty}\left(\frac{t^{0.7}}{0.7} - \frac{1}{0.7}\right) = \infty.$$

Since this improper integral is divergent, the series $\displaystyle\sum_{n=1}^{\infty} n^{-0.3}$ is also divergent by the Integral Test.

5. The function $f(x) = \dfrac{2}{5x-1}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{2}{5x-1}\,dx = \lim_{t\to\infty} \int_1^t \frac{2}{5x-1}\,dx = \lim_{t\to\infty} \left[\frac{2}{5}\ln(5x-1)\right]_1^t = \lim_{t\to\infty}\left[\frac{2}{5}\ln(5t-1) - \frac{2}{5}\ln 4\right] = \infty.$$

Since this improper integral is divergent, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{5n-1}$ is also divergent by the Integral Test.

6. The function $f(x) = \dfrac{1}{(3x-1)^4}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{1}{(3x-1)^4}\,dx = \lim_{t\to\infty} \int_1^t (3x-1)^{-4}\,dx = \lim_{t\to\infty} \left[\frac{1}{(-3)3}(3x-1)^{-3}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{9(3t-1)^3} + \frac{1}{9\cdot 2^3}\right] = \frac{1}{72}.$$

Since this improper integral is convergent, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n-1)^4}$ is also convergent by the Integral Test.

12. $\sum\limits_{n=3}^{\infty} n^{-0.9999} = \sum\limits_{n=3}^{\infty} \frac{1}{n^{0.9999}}$ is a p-series with $p = 0.9999 \le 1$, so it diverges by (1). The fact that the series begins with $n = 3$ is irrelevant when determining convergence.

13. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots = \sum\limits_{n=1}^{\infty} \frac{1}{n^3}$. This is a p-series with $p = 3 > 1$, so it converges by (1).

14. $\frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \cdots = \sum\limits_{n=1}^{\infty} \frac{1}{2n+3}$. The function $f(x) = \frac{1}{2x+3}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{1}{2x+3}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{2x+3}\,dx = \lim_{t\to\infty} \left[\tfrac{1}{2}\ln(2x+3)\right]_1^t = \lim_{t\to\infty} \left[\tfrac{1}{2}\ln(2t+3) - \tfrac{1}{2}\ln 5\right] = \infty, \text{ so the series}$$

$\sum\limits_{n=1}^{\infty} \frac{1}{2n+3}$ diverges.

15. $\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \frac{1}{19} + \cdots = \sum\limits_{n=1}^{\infty} \frac{1}{4n-1}$. The function $f(x) = \frac{1}{4x-1}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{1}{4x-1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{4x-1}\,dx = \lim_{t\to\infty} \left[\tfrac{1}{4}\ln(4x-1)\right]_1^t = \lim_{t\to\infty} \left[\tfrac{1}{4}\ln(4t-1) - \tfrac{1}{4}\ln 3\right] = \infty, \text{ so the series}$$

$\sum\limits_{n=1}^{\infty} \frac{1}{4n-1}$ diverges.

16. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots = \sum\limits_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum\limits_{n=1}^{\infty} \frac{1}{n^{3/2}}$. This is a p-series with $p = \frac{3}{2} > 1$, so it converges by (1).

17. $\sum\limits_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2} = \sum\limits_{n=1}^{\infty} \left(\frac{\sqrt{n}}{n^2} + \frac{4}{n^2}\right) = \sum\limits_{n=1}^{\infty} \frac{1}{n^{3/2}} + \sum\limits_{n=1}^{\infty} \frac{4}{n^2}$. $\sum\limits_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series with $p = \frac{3}{2} > 1$. $\sum\limits_{n=1}^{\infty} \frac{4}{n^2} = 4\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is a constant multiple of a convergent p-series with $p = 2 > 1$, so it converges. The sum of two convergent series is convergent, so the original series is convergent.

18. The function $f(x) = \frac{\sqrt{x}}{1 + x^{3/2}}$ is continuous and positive on $[1, \infty)$.

$$f'(x) = \frac{(1+x^{3/2})\left(\frac{1}{2}x^{-1/2}\right) - x^{1/2}\left(\frac{3}{2}x^{1/2}\right)}{(1+x^{3/2})^2} = \frac{\frac{1}{2}x^{-1/2} + \frac{1}{2}x - \frac{3}{2}x}{(1+x^{3/2})^2} = \frac{1 - 2x^{3/2}}{2\sqrt{x}(1+x^{3/2})^2} < 0 \text{ for } x \ge 1, \text{ so } f \text{ is}$$

decreasing on $[1, \infty)$, and the Integral Test applies.

$$\begin{aligned} \int_1^{\infty} \frac{\sqrt{x}}{1+x^{3/2}}\,dx &= \lim_{t\to\infty} \int_1^t \frac{\sqrt{x}}{1+x^{3/2}}\,dx = \lim_{t\to\infty} \left[\tfrac{2}{3}\ln(1+x^{3/2})\right]_1^t \qquad \left[\begin{matrix}\text{substitution} \\ \text{with } u = 1 + x^{3/2}\end{matrix}\right] \\ &= \lim_{t\to\infty} \left[\tfrac{2}{3}\ln(1+t^{3/2}) - \tfrac{2}{3}\ln 2\right] = \infty, \end{aligned}$$

so the series $\sum\limits_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^{3/2}}$ diverges.

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19. The function $f(x) = \dfrac{1}{x^2+4}$ is continuous, positive, and decreasing on $[1, \infty)$, so we can apply the Integral Test.

$$\int_1^\infty \frac{1}{x^2+4}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+4}\,dx = \lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_1^t = \frac{1}{2}\lim_{t\to\infty}\left[\tan^{-1}\left(\frac{t}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right]$$
$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)\right]$$

Therefore, the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+4}$ converges.

20. The function $f(x) = \dfrac{1}{x^2+2x+2}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^\infty \frac{1}{x^2+2x+2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(x+1)^2+1}\,dx = \lim_{t\to\infty}\Big[\arctan(x+1)\Big]_1^t$$
$$= \lim_{t\to\infty}[\arctan(t+1) - \arctan 2] = \frac{\pi}{2} - \arctan 2,$$

so the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+2n+2}$ converges.

21. The function $f(x) = \dfrac{x^3}{x^4+4}$ is continuous and positive on $[2, \infty)$, and is also decreasing since

$$f'(x) = \frac{(x^4+4)(3x^2) - x^3(4x^3)}{(x^4+4)^2} = \frac{12x^2 - x^6}{(x^4+4)^2} = \frac{x^2(12-x^4)}{(x^4+4)^2} < 0 \text{ for } x > \sqrt[4]{12} \approx 1.86, \text{ so we can use the}$$

Integral Test on $[2, \infty)$.

$$\int_2^\infty \frac{x^3}{x^4+4}\,dx = \lim_{t\to\infty}\int_2^t \frac{x^3}{x^4+4}\,dx = \lim_{t\to\infty}\left[\tfrac{1}{4}\ln(x^4+4)\right]_2^t = \lim_{t\to\infty}\left[\tfrac{1}{4}\ln(t^4+4) - \tfrac{1}{4}\ln 20\right] = \infty, \text{ so the series}$$

$\sum\limits_{n=2}^{\infty} \dfrac{n^3}{n^4+4}$ diverges, and it follows that $\sum\limits_{n=1}^{\infty} \dfrac{n^3}{n^4+4}$ diverges as well.

22. The function $f(x) = \dfrac{3x-4}{x^2-2x} = \dfrac{2}{x} + \dfrac{1}{x-2}$ [by partial fractions] is continuous, positive, and decreasing on $[3, \infty)$ since it is the sum of two such functions, so we can apply the Integral Test.

$$\int_3^\infty \frac{3x-4}{x^2-x}\,dx = \lim_{t\to\infty}\int_3^t \left[\frac{2}{x} + \frac{1}{x-2}\right]dx = \lim_{t\to\infty}\Big[2\ln x + \ln(x-2)\Big]_3^t = \lim_{t\to\infty}[2\ln t + \ln(t-2) - 2\ln 3] = \infty.$$

The integral is divergent, so the series $\sum\limits_{n=3}^{\infty} \dfrac{3n-4}{n^2-n}$ is divergent.

23. $f(x) = \dfrac{1}{x\ln x}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = -\dfrac{1+\ln x}{x^2(\ln x)^2} < 0$ for $x > 2$, so we can use the Integral Test. $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{t\to\infty}[\ln(\ln x)]_2^t = \lim_{t\to\infty}[\ln(\ln t) - \ln(\ln 2)] = \infty$, so the series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\ln n}$ diverges.

24. The function $f(x) = \dfrac{\ln x}{x^2}$ is continuous and positive on $[2, \infty)$, and also decreasing since

$$f'(x) = \frac{x^2(1/x) - (\ln x)(2x)}{(x^2)^2} = \frac{x - 2x\ln x}{x^4} = \frac{1-2\ln x}{x^3} < 0 \text{ for } x > e^{1/2} \approx 1.65, \text{ so we can use the Integral Test}$$