

HW #4, SECTION 11.3 Solutions

34. If $p \leq 0$, $\lim_{n \rightarrow \infty} \frac{\ln n}{n^p} = \infty$ and the series diverges, so assume $p > 0$. $f(x) = \frac{\ln x}{x^p}$ is positive and continuous and $f'(x) < 0$

for $x > e^{1/p}$, so f is eventually decreasing and we can use the Integral Test. Integration by parts gives

$$\int_1^{\infty} \frac{\ln x}{x^p} dx = \lim_{t \rightarrow \infty} \left[\frac{x^{1-p} [(1-p) \ln x - 1]}{(1-p)^2} \right]_1^t \quad (\text{for } p \neq 1) = \frac{1}{(1-p)^2} \left[\lim_{t \rightarrow \infty} t^{1-p} [(1-p) \ln t - 1] + 1 \right], \text{ which exists}$$

whenever $1-p < 0 \Leftrightarrow p > 1$. Thus, $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ converges $\Leftrightarrow p > 1$.

35. Since this is a p -series with $p = x$, $\zeta(x)$ is defined when $x > 1$. Unless specified otherwise, the domain of a function f is the set of real numbers x such that the expression for $f(x)$ makes sense and defines a real number. So, in the case of a series, it's the set of real numbers x such that the series is convergent.

36. (a) $\sum_{n=2}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{1^2}$ [subtract a_1] $= \frac{\pi^2}{6} - 1$

(b) $\sum_{n=3}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=4}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \right) = \frac{\pi^2}{6} - \frac{49}{36}$

(c) $\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} \left(\frac{\pi^2}{6} \right) = \frac{\pi^2}{24}$

37. (a) $\sum_{n=1}^{\infty} \left(\frac{3}{n} \right)^4 = \sum_{n=1}^{\infty} \frac{81}{n^4} = 81 \sum_{n=1}^{\infty} \frac{1}{n^4} = 81 \left(\frac{\pi^4}{90} \right) = \frac{9\pi^4}{10}$

(b) $\sum_{k=5}^{\infty} \frac{1}{(k-2)^4} = \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots = \sum_{k=3}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90} - \left(\frac{1}{1^4} + \frac{1}{2^4} \right)$ [subtract a_1 and a_2] $= \frac{\pi^4}{90} - \frac{17}{16}$

38. (a) $f(x) = 1/x^4$ is positive and continuous and $f'(x) = -4/x^5$ is negative for $x > 0$, and so the Integral Test applies.

$\sum_{n=1}^{\infty} \frac{1}{n^4} \approx s_{10} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{10^4} \approx 1.082037$.

$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^4} dx = \lim_{t \rightarrow \infty} \left[\frac{1}{-3x^3} \right]_{10}^t = \lim_{t \rightarrow \infty} \left(-\frac{1}{3t^3} + \frac{1}{3(10)^3} \right) = \frac{1}{3000}$, so the error is at most 0.0003.

(b) $s_{10} + \int_{11}^{\infty} \frac{1}{x^4} dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^4} dx \Rightarrow s_{10} + \frac{1}{3(11)^3} \leq s \leq s_{10} + \frac{1}{3(10)^3} \Rightarrow$

$1.082037 + 0.000250 = 1.082287 \leq s \leq 1.082037 + 0.000333 = 1.082370$, so we get $s \approx 1.08233$ with error ≤ 0.00005 .

(c) The estimate in part (b) is $s \approx 1.08233$ with error ≤ 0.00005 . The exact value given in Exercise 37 is $\pi^4/90 \approx 1.082323$. The difference is less than 0.00001.

(d) $R_n \leq \int_n^{\infty} \frac{1}{x^4} dx = \frac{1}{3n^3}$. So $R_n < 0.00001 \Rightarrow \frac{1}{3n^3} < \frac{1}{10^5} \Rightarrow 3n^3 > 10^5 \Rightarrow n > \sqrt[3]{(10)^5/3} \approx 32.2$,

that is, for $n > 32$.

39. (a) $f(x) = \frac{1}{x^2}$ is positive and continuous and $f'(x) = -\frac{2}{x^3}$ is negative for $x > 0$, and so the Integral Test applies.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx s_{10} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{10^2} \approx 1.549768.$$

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} \left[\frac{-1}{x} \right]_{10}^t = \lim_{t \rightarrow \infty} \left(-\frac{1}{t} + \frac{1}{10} \right) = \frac{1}{10}, \text{ so the error is at most } 0.1.$$

$$(b) s_{10} + \int_{11}^{\infty} \frac{1}{x^2} dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^2} dx \Rightarrow s_{10} + \frac{1}{11} \leq s \leq s_{10} + \frac{1}{10} \Rightarrow$$

$1.549768 + 0.090909 = 1.640677 \leq s \leq 1.549768 + 0.1 = 1.649768$, so we get $s \approx 1.64522$ (the average of 1.640677 and 1.649768) with error ≤ 0.005 (the maximum of $1.649768 - 1.64522$ and $1.64522 - 1.640677$, rounded up).

- (c) The estimate in part (b) is $s \approx 1.64522$ with error ≤ 0.005 . The exact value given in Exercise 36 is $\pi^2/6 \approx 1.644934$. The difference is less than 0.0003.

$$(d) R_n \leq \int_n^{\infty} \frac{1}{x^2} dx = \frac{1}{n}. \text{ So } R_n < 0.001 \text{ if } \frac{1}{n} < \frac{1}{1000} \Leftrightarrow n > 1000.$$

40. $\sum_{n=1}^{\infty} n e^{-2n}$. $f(x) = x e^{-2x}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies. Using (2),

$$\begin{aligned} R_n &\leq \int_n^{\infty} x e^{-2x} dx = \lim_{t \rightarrow \infty} \left(\left[-\frac{1}{2} x e^{-2x} \right]_n^t + \int_n^t \frac{1}{2} e^{-2x} dx \right) \quad \left[\begin{array}{l} \text{using parts with} \\ u = x, dv = e^{-2x} dx \end{array} \right] \\ &= \lim_{t \rightarrow \infty} \left(\frac{-t}{2e^{2t}} + \frac{n}{2e^{2n}} - \frac{1}{4e^{2t}} + \frac{1}{4e^{2n}} \right) \stackrel{H}{=} 0 + \frac{n}{2e^{2n}} - 0 + \frac{1}{4e^{2n}} = \frac{2n+1}{4e^{2n}} \end{aligned}$$

To be correct to four decimal places, we want $\frac{2n+1}{4e^{2n}} \leq \frac{5}{10^5}$. This inequality is true for $n = 6$.

$$s_6 = \sum_{n=1}^6 \frac{n}{e^{2n}} = \frac{1}{e^2} + \frac{2}{e^4} + \frac{3}{e^6} + \frac{4}{e^8} + \frac{5}{e^{10}} + \frac{6}{e^{12}} \approx 0.1810.$$

41. $\sum_{n=1}^{\infty} (2n+1)^{-6}$. $f(x) = 1/(2x+1)^6$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\text{Using (2), } R_n \leq \int_n^{\infty} (2x+1)^{-6} dx = \lim_{t \rightarrow \infty} \left[\frac{-1}{10(2x+1)^5} \right]_n^t = \frac{1}{10(2n+1)^5}. \text{ To be correct to five decimal places,}$$

$$\text{we want } \frac{1}{10(2n+1)^5} \leq \frac{5}{10^6} \Leftrightarrow (2n+1)^5 \geq 20,000 \Leftrightarrow n \geq \frac{1}{2}(\sqrt[5]{20,000} - 1) \approx 3.12, \text{ so use } n = 4.$$

$$s_4 = \sum_{n=1}^4 \frac{1}{(2n+1)^6} = \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} \approx 0.001446 \approx 0.00145.$$

42. $f(x) = \frac{1}{x(\ln x)^2}$ is positive and continuous and $f'(x) = -\frac{\ln x + 2}{x^2(\ln x)^3}$ is negative for $x > 1$, so the Integral Test applies.

$$\text{Using (2), we need } 0.01 > \int_n^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{t \rightarrow \infty} \left[\frac{-1}{\ln x} \right]_n^t = \frac{1}{\ln n}. \text{ This is true for } n > e^{100}, \text{ so we would have to add this}$$

many terms to find the sum of the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ to within 0.01, which would be problematic because

$$e^{100} \approx 2.7 \times 10^{43}.$$