HW #4, SECTION 11.4 SOLUTIONS

(c) We have shown that $\{t_n\}$ is decreasing and that $t_n > 0$ for all n. Thus, $0 < t_n \le t_1 = 1$, so $\{t_n\}$ is a bounded monotonic sequence, and hence converges by the Monotonic Sequence Theorem.

47. $b^{\ln n} = \left(e^{\ln b}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln b} = n^{\ln b} = \dfrac{1}{n^{-\ln b}}$. $\sum\limits_{n=1}^{\infty} b^{\ln n}$ is a p-series, which converges for all b such that $-\ln b > 1 \iff \ln b < -1 \iff b < e^{-1} \iff b < 1/e$ [with $b > 0$].

48. For the series $\sum\limits_{n=1}^{\infty}\left(\dfrac{c}{n} - \dfrac{1}{n+1}\right)$,

$$s_n = \sum_{i=1}^{n}\left(\frac{c}{i} - \frac{1}{i+1}\right) = \left(\frac{c}{1} - \frac{1}{2}\right) + \left(\frac{c}{2} - \frac{1}{3}\right) + \left(\frac{c}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{c}{n} - \frac{1}{n+1}\right)$$
$$= \frac{c}{1} + \frac{c-1}{2} + \frac{c-1}{3} + \frac{c-1}{4} + \cdots + \frac{c-1}{n} - \frac{1}{n+1} = c + (c-1)\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right) - \frac{1}{n+1}$$

Thus, $\sum\limits_{n=1}^{\infty}\left(\dfrac{c}{n} - \dfrac{1}{n+1}\right) = \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty}\left[c + (c-1)\sum\limits_{i=2}^{n}\dfrac{1}{i} - \dfrac{1}{n+1}\right]$. Since a constant multiple of a divergent series is divergent, the last limit exists only if $c - 1 = 0$, so the original series converges only if $c = 1$.

11.4 The Comparison Tests

1. (a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent. (See the discusssion preceding the box titled "The Limit Comparison Test.")

(b) If $a_n < b_n$ for all n, then $\sum a_n$ is convergent. [This is part (i) of the Direct Comparison Test.]

2. (a) If $a_n > b_n$ for all n, then $\sum a_n$ is divergent. [This is part (ii) of the Direct Comparison Test.]

(b) We cannot say anything about $\sum a_n$. If $a_n < b_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ could be convergent or divergent.

3. (a) $\dfrac{n}{n^3+5} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$ for all $n \ge 2$. $\sum\limits_{n=2}^{\infty}\dfrac{1}{n^2}$ converges because it is a p-series with $p = 2 > 1$, so $\sum\limits_{n=2}^{\infty}\dfrac{n}{n^3+5}$ converges by part (i) of the Direct Comparison Test.

(b) Use the Limit Comparison Test with $a_n = \dfrac{n}{n^3-5}$ and $b_n = \dfrac{1}{n^2}$:

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n}{n^3-5}\cdot\frac{n^2}{1} = \lim_{n\to\infty}\frac{n^3}{n^3\left(1-5/n^3\right)} = \lim_{n\to\infty}\frac{1}{1-5/n^3} = \frac{1}{1-0} = 1 > 0$$

Since $\sum\limits_{n=2}^{\infty}\dfrac{1}{n^2}$ is a convergent (partial) p-series [$p = 2 > 1$], the series $\sum\limits_{n=2}^{\infty}\dfrac{n}{n^3-5}$ also converges.

4. (a) $\dfrac{n^2+n}{n^3-2} > \dfrac{n^2}{n^3-2} > \dfrac{n^2}{n^3} = \dfrac{1}{n}$ for all $n \ge 2$. $\sum\limits_{n=2}^{\infty}\dfrac{1}{n}$ diverges because it is a (partial) p-series with $p = 1 \le 1$, so $\sum\limits_{n=2}^{\infty}\dfrac{n^2+n}{n^3-2}$ diverges by part (ii) of the Direct Comparison Test.

(b) Use the Limit Comparison Test with $a_n = \dfrac{n^2 - n}{n^3 + 2}$ and $b_n = \dfrac{1}{n}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2 - n}{n^3 + 2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^3 - n^2}{n^3 + 2} = \lim_{n\to\infty} \frac{1 - 1/n}{1 + 2/n^3} = \frac{1 - 0}{1 + 0} = 1 > 0$$

Since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n}$ is a divergent (partial) p-series $[p = 1 \le 1]$, the series $\sum\limits_{n=2}^{\infty} \dfrac{n^2 - n}{n^3 + 2}$ also diverges.

5. An inequality can be used to show that a series converges if its general term can be shown to be less than or equal to the general term of a known convergent series. The only inequality that satisfies this condition is given in part (c) since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ is a convergent p-series $[p = 2 > 1]$.

6. An inequality can be used to show that a series diverges if its general term can be shown to be greater than or equal to the general term of a known divergent series. The only inequality that satisfies this condition is given in part (c) since $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n} = \dfrac{1}{2} \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ is half of the harmonic series, which is divergent.

7. $\dfrac{1}{n^3 + 8} < \dfrac{1}{n^3}$ for all $n \ge 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3 + 8}$ converges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$, which converges because it is a p-series with $p = 3 > 1$.

8. $\dfrac{1}{\sqrt{n} - 1} > \dfrac{1}{\sqrt{n}}$ for all $n \ge 2$, so $\sum\limits_{n=2}^{\infty} \dfrac{1}{\sqrt{n} - 1}$ diverges by direct comparison with $\sum\limits_{n=2}^{\infty} \dfrac{1}{\sqrt{n}}$, which diverges because it is a p-series with $p = \frac{1}{2} \le 1$.

9. $\dfrac{n + 1}{n\sqrt{n}} > \dfrac{n}{n\sqrt{n}} = \dfrac{1}{\sqrt{n}}$ for all $n \ge 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n + 1}{n\sqrt{n}}$ diverges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$, which diverges because it is a p-series with $p = \frac{1}{2} \le 1$.

10. $\dfrac{n - 1}{n^3 + 1} < \dfrac{n}{n^3 + 1} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$ for all $n \ge 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n - 1}{n^3 + 1}$ converges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

11. $\dfrac{9^n}{3 + 10^n} < \dfrac{9^n}{10^n} = \left(\dfrac{9}{10}\right)^n$ for all $n \ge 1$. $\sum\limits_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$ is a convergent geometric series $\left(|r| = \frac{9}{10} < 1\right)$, so $\sum\limits_{n=1}^{\infty} \dfrac{9^n}{3 + 10^n}$ converges by the Direct Comparison Test.

12. $\dfrac{6^n}{5^n - 1} > \dfrac{6^n}{5^n} = \left(\dfrac{6}{5}\right)^n$ for all $n \ge 1$. $\sum\limits_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$ is a divergent geometric series $\left(|r| = \frac{6}{5} > 1\right)$, so $\sum\limits_{n=1}^{\infty} \dfrac{6^n}{5^n - 1}$ diverges by the Direct Comparison Test.

13. For $n \ge 2$, $\ln n < n$, so $\dfrac{1}{\ln n} > \dfrac{1}{n}$. Thus, $\sum\limits_{n=2}^{\infty} \dfrac{1}{\ln n}$ diverges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$, which diverges because it is a p-series with $p = 1 \le 1$ (the harmonic series).