HW #5, Section 11.5 SOLUTIONS Sec 11.5

21. (a) A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent. If a series is absolutely convergent, then it is convergent.

(b) A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent; that is, if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

(c) Suppose the series of positive terms $\sum_{n=1}^{\infty} b_n$ converges. Then $\sum |(-1)^n b_n| = \sum |b_n| = \sum b_n$ also converges, so $\sum_{n=1}^{\infty} (-1)^n b_n$ is absolutely convergent (and therefore convergent).

22. $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^4}\right| = \sum_{n=1}^{\infty} \frac{1}{n^4}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a convergent p-series $[p = 4 > 1]$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ is absolutely convergent.

23. $b_n = \frac{1}{\sqrt[3]{n^2}} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing for $n \geq 1$, and $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2}} = 0$, so the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt[3]{n^2}}$ converges by the Alternating Series Test. Also, observe that $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{\sqrt[3]{n^2}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ is divergent since it is a p-series with $p = \frac{2}{3} \leq 1$. Thus, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt[3]{n^2}}$ is conditionally convergent.

24. Since $\lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+1/n^2} = \frac{1}{1+0} = 1 \neq 0$ and $\lim_{n\to\infty} (-1)^{n+1}$ does not exist, $\lim_{n\to\infty} (-1)^{n+1} \frac{n^2}{n^2+1}$ does not exist, so the series $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{n^2}{n^2+1}$ diverges by the Test for Divergence.

25. $b_n = \frac{1}{5n+1} > 0$ for $n \geq 1$, $\{b_n\}$ is decreasing for $n \geq 1$, and $\lim_{n\to\infty} b_n = 0$, so $\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}$ converges by the Alternating Series Test. To determine absolute convergence, choose $a_n = \frac{1}{n}$ to get

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n}{1/(5n+1)} = \lim_{n\to\infty} \frac{5n+1}{n} = 5 > 0$, so $\sum_{n=1}^{\infty} \frac{1}{5n+1}$ diverges by the Limit Comparison Test with the harmonic series. Thus, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}$ is conditionally convergent.

26. $\sum_{n=1}^{\infty} \frac{-n}{n^2+1} = -\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. Use the Limit Comparison Test with $a_n = \frac{n}{n^2+1}$ and $b_n = \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n/(n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+1/n^2} = \frac{1}{1+0} = 1 > 0$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series $[p = 1 \leq 1]$, the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ also diverges, and hence, the negative of this series, $\sum_{n=1}^{\infty} \frac{-n}{n^2+1}$, diverges.

27. $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2+1}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$. Since $\frac{1}{n^2+1} < \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series $[p = 2 > 1]$, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent by the Direct Comparison Test. Thus, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ is absolutely convergent.

28. $0 < \left|\frac{\sin n}{2^n}\right| < \frac{1}{2^n}$ for $n \geq 1$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series $[r = \frac{1}{2} < 1]$, so $\sum_{n=1}^{\infty} \left|\frac{\sin n}{2^n}\right|$ converges by direct comparison and the series $\sum_{n=1}^{\infty} \frac{\sin n}{2^n}$ is absolutely convergent.

29. $0 < \left|\frac{1+2\sin n}{n^3}\right| < \frac{3}{n^3}$ for $n \geq 1$ and $3\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a constant times a convergent p-series $[p = 3 > 1]$, so $\sum_{n=1}^{\infty} \left|\frac{1+2\sin n}{n^3}\right|$ converges by direct comparison and the series $\sum_{n=1}^{\infty} \frac{1+2\sin n}{n^3}$ is absolutely convergent.

30. $b_n = \frac{n}{n^2+4} > 0$ for $n \geq 1$, $\{b_n\}$ is decreasing for $n \geq 2$, and $\lim_{n\to\infty} b_n = 0$, so $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$ converges by the Alternating Series Test. To determine absolute convergence, choose $a_n = \frac{1}{n}$ to get

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n}{n/(n^2+4)} = \lim_{n\to\infty} \frac{n^2+4}{n^2} = \lim_{n\to\infty} \frac{1+4/n^2}{1} = 1 > 0,$$ so $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ diverges by the Limit Comparison Test with the harmonic series. Thus, the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$ is conditionally convergent.

31. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges by the Alternating Series Test since $\lim_{n\to\infty} \frac{1}{\ln n} = 0$ and $\left\{\frac{1}{\ln n}\right\}$ is decreasing. Now $\ln n < n$, so $\frac{1}{\ln n} > \frac{1}{n}$, and since $\sum_{n=2}^{\infty} \frac{1}{n}$ is the divergent (partial) harmonic series, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by the Direct Comparison Test. Thus, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is conditionally convergent.

32. $b_n = \frac{n}{\sqrt{n^3+2}} > 0$ for $n \geq 1$, $\{b_n\}$ is decreasing for $n \geq 2$, and $\lim_{n\to\infty} b_n = 0$, so $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$ converges by the Alternating Series Test. To determine absolute convergence, choose $a_n = \frac{1}{\sqrt{n}}$ to get

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{1}{\sqrt{n}} \cdot \frac{\sqrt{n^3+2}}{n}\right) = \lim_{n\to\infty} \frac{\sqrt{n^3+2}}{\sqrt{n^3}} = \lim_{n\to\infty} \sqrt{1+\frac{2}{n^3}} = 1 > 0,$$ so $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+2}}$ diverges by limit comparision with the divergent p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $\left[p = \frac{1}{2} \leq 1\right]$. Thus, $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$ is conditionally convergent.

33. $a_n = \frac{\cos n\pi}{3n+2} = (-1)^n \frac{1}{3n+2} = (-1)^n b_n$. $\{b_n\}$ is decreasing for $n \geq 1$, and $\lim_{n\to\infty} b_n = 0$, so $\sum_{n=1}^{\infty} \frac{\cos n\pi}{3n+2}$ converges by the Alternating Series Test. To determine absolute convergence, use the Limit Comparison Test with $a_n = \frac{1}{n}$:

[continued]

37. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^6} < \frac{1}{n^6}$ and (ii) $\lim_{n\to\infty} \frac{1}{n^6} = 0$, so the series is convergent. Now $b_5 = \frac{1}{5^6} = 0.000064 > 0.00005$ and $b_6 = \frac{1}{6^6} \approx 0.00002 < 0.00005$, so by the Alternating Series Estimation Theorem, $n = 5$. (That is, since the 6th term is less than the desired error, we need to add the first 5 terms to get the sum to the desired accuracy.)

38. The series $\sum_{n=1}^{\infty} \frac{(-\frac{1}{3})^n}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n3^n}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)3^{n+1}} < \frac{1}{n3^n}$ and (ii) $\lim_{n\to\infty} \frac{1}{n3^n} = 0$, so the series is convergent. Now $b_5 = \frac{1}{5 \cdot 3^5} \approx 0.0008 > 0.0005$ and $b_6 = \frac{1}{6 \cdot 3^6} \approx 0.0002 < 0.0005$, so by the Alternating Series Estimation Theorem, $n = 5$. (That is, since the 6th term is less than the desired error, we need to add the first 5 terms to get the sum to the desired accuracy.)

39. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 2^n}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^2 2^{n+1}} < \frac{1}{n^2 2^n}$ and (ii) $\lim_{n\to\infty} \frac{1}{n^2 2^n} = 0$, so the series is convergent. Now $b_5 = \frac{1}{5^2 2^5} = 0.00125 > 0.0005$ and $b_6 = \frac{1}{6^2 2^6} \approx 0.0004 < 0.0005$, so by the Alternating Series Estimation Theorem, $n = 5$. (That is, since the 6th term is less than the desired error, we need to add the first 5 terms to get the sum to the desired accuracy.)

40. The series $\sum_{n=1}^{\infty} \left(-\frac{1}{n}\right)^n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^n}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^{n+1}} < \frac{1}{n^n}$ and (ii) $\lim_{n\to\infty} \frac{1}{n^n} = 0$, so the series is convergent. Now $b_5 = \frac{1}{5^5} = 0.00032 > 0.00005$ and $b_6 = \frac{1}{6^6} \approx 0.00002 < 0.00005$, so by the Alternating Series Estimation Theorem, $n = 5$. (That is, since the 6th term is less than the desired error, we need to add the first 5 terms to get the sum to the desired accuracy.)

41. $b_4 = \frac{1}{8!} = \frac{1}{40{,}320} \approx 0.000\,025$, so

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \approx s_3 = \sum_{n=1}^{3} \frac{(-1)^n}{(2n)!} = -\frac{1}{2} + \frac{1}{24} - \frac{1}{720} \approx -0.459\,722$$

Adding b_4 to s_3 does not change the fourth decimal place of s_3, so by the Alternating Series Estimation Theorem, the sum of the series, correct to four decimal places, is -0.4597.

42. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} \approx s_9 = \frac{1}{1^6} - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \frac{1}{6^6} + \frac{1}{7^6} - \frac{1}{8^6} + \frac{1}{9^6} \approx 0.985\,552$. Subtracting $b_{10} = 1/10^6$ from s_9 does not change the fourth decimal place of s_9, so by the Alternating Series Estimation Theorem, the sum of the series, correct to four decimal places, is 0.9856.

43. $\sum_{n=1}^{\infty} (-1)^n n e^{-2n} \approx s_5 = -\frac{1}{e^2} + \frac{2}{e^4} - \frac{3}{e^6} + \frac{4}{e^8} - \frac{5}{e^{10}} \approx -0.105\,025$. Adding $b_6 = 6/e^{12} \approx 0.000\,037$ to s_5 does not change the fourth decimal place of s_5, so by the Alternating Series Estimation Theorem, the sum of the series, correct to four decimal places, is -0.1050.