HW #5, SECTION 11.8 SOLUTIONS

(b) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (a), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers; that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

3. If $a_n = \dfrac{x^n}{n}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{n+1}\cdot\dfrac{n}{x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{nx}{n+1}\right| = \lim\limits_{n\to\infty}\left(\dfrac{1}{1+1/n}\,|x|\right) = |x|$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{x^n}{n}$ converges when $|x| < 1$, so the radius of convergence is $R = 1$. Now we'll check the endpoints, that is, $x = \pm 1$. When $x = 1$, the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$ diverges since it is the harmonic series. When $x = -1$, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n}$ converges by the Alternating Series Test. Thus, the interval of convergence is $[-1, 1)$.

4. If $a_n = (-1)^n n x^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)x^{n+1}}{(-1)^n\,nx^n}\right| = \lim_{n\to\infty}\left|(-1)\frac{n+1}{n}x\right| = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)|x|\right] = |x|.$$

By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}(-1)^n nx^n$ converges when $|x| < 1$, so the radius of convergence $R = 1$. Now we'll check the endpoints, that is, $x = \pm 1$. Both series $\sum\limits_{n=1}^{\infty}(-1)^n n(\pm 1)^n = \sum\limits_{n=1}^{\infty}(\mp 1)^n n$ diverge by the Test for Divergence since $\lim\limits_{n\to\infty}|(\mp 1)^n n| = \infty$. Thus, the interval of convergence is $I = (-1, 1)$.

5. If $a_n = \sqrt{n}\,x^n$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{\sqrt{n+1}\,x^{n+1}}{\sqrt{n}\,x^n}\right| = \lim\limits_{n\to\infty}\left|\sqrt{\dfrac{n+1}{n}}\,x\right| = \lim\limits_{n\to\infty}\sqrt{1+\dfrac{1}{n}}\,|x| = |x|$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\sqrt{n}\,x^n$ converges when $|x| < 1$, so $R = 1$. When $x = \pm 1$, both series $\sum\limits_{n=1}^{\infty}\sqrt{n}\,(\pm 1)^n$ diverge by the Test for Divergence since $\lim\limits_{n\to\infty}|\sqrt{n}\,(\pm 1)^n| = \infty$. Thus, the interval of convergence is $(-1, 1)$.

6. If $a_n = \dfrac{(-1)^n x^n}{\sqrt[3]{n}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{\sqrt[3]{n+1}}\cdot\frac{\sqrt[3]{n}}{(-1)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)x\sqrt[3]{n}}{\sqrt[3]{n+1}}\right| = \lim_{n\to\infty}\sqrt[3]{\frac{1}{1+1/n}}\,|x| = |x|.$$

By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n x^n}{\sqrt[3]{n}}$ converges when $|x| < 1$, so $R = 1$. When $x = 1$, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{\sqrt[3]{n}}$ converges by the Alternating Series Test. When $x = -1$, the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt[3]{n}}$ diverges since it is a p-series $\left(p = \frac{1}{3} \le 1\right)$. Thus, the interval of convergence is $(-1, 1]$.

7. If $a_n = \dfrac{n}{5^n}\,x^n$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)\,x^{n+1}}{5^{n+1}}\cdot\dfrac{5^n}{n\,x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{n+1}{5n}\,x\right| = \lim\limits_{n\to\infty}\left(\dfrac{1}{5}+\dfrac{1}{5n}\right)|x| = \dfrac{|x|}{5}$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{n}{5^n}\,x^n$ converges when $\dfrac{|x|}{5} < 1 \;\Leftrightarrow\; |x| < 5$, so $R = 5$. When $x = \pm 5$, both series

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$\sum_{n=1}^{\infty} \frac{n(\pm 5)^n}{5^n} = \sum_{n=1}^{\infty} (\pm 1)^n n$ diverge by the Test for Divergence since $\lim_{n\to\infty} |(\pm 1)^n n| = \infty$. Thus, the interval of convergence is $(-5, 5)$.

8. If $a_n = \frac{5^n}{n} x^n$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{5^{n+1}x^{n+1}}{(n+1)} \cdot \frac{n}{5^n x^n}\right| = \lim_{n\to\infty} \left|\frac{5n}{n+1} x\right| = \lim_{n\to\infty} \left(\frac{5}{1+1/n}|x|\right) = 5|x|$. By the Ratio Test, the series $\sum_{n=2}^{\infty} \frac{5^n}{n} x^n$ converges when $5|x| < 1 \iff |x| < \frac{1}{5}$, so $R = \frac{1}{5}$. When $x = \frac{1}{5}$, the series $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges since it is the (partial) harmonic series. When $x = -\frac{1}{5}$, the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n}$ converges by the Alternating Series Test. Thus, the interval of convergence is $\left[-\frac{1}{5}, \frac{1}{5}\right)$.

9. If $a_n = \frac{x^n}{n\,3^n}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)\,3^{n+1}} \cdot \frac{n\,3^n}{x^n}\right| = \lim_{n\to\infty} \left|\frac{n}{3(n+1)} x\right| = \lim_{n\to\infty} \left(\frac{1}{3+3/n}|x|\right) = \frac{|x|}{3}$. By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{x^n}{n\,3^n}$ converges when $\frac{|x|}{3} < 1 \iff |x| < 3$, so $R = 3$. When $x = 3$, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges since it is the harmonic series. When $x = -3$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the Alternating Series Test. Thus, the interval of convergence is $[-3, 3)$.

10. If $a_n = \frac{n}{n+1} x^n$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)\,x^{n+1}}{(n+1)+1} \cdot \frac{n+1}{n\,x^n}\right| = \lim_{n\to\infty} \left|\frac{n^2+2n+1}{n^2+2n} x\right| = \lim_{n\to\infty} \left(\frac{1+2/n+1/n^2}{1+2/n}|x|\right) = |x|.$$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{n}{n+1} x^n$ converges when $|x| < 1$, so $R = 1$. When $x = \pm 1$, both series $\sum_{n=1}^{\infty} \frac{n(\pm 1)^n}{n+1}$ diverge by the Test for Divergence since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$ and $\lim_{n\to\infty} \frac{n(-1)^n}{n+1}$ does not exist. Thus, the interval of convergence is $(-1, 1)$.

11. If $a_n = \frac{x^n}{2n-1}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{2n+1} \cdot \frac{2n-1}{x^n}\right| = \lim_{n\to\infty} \left(\frac{2n-1}{2n+1}|x|\right) = \lim_{n\to\infty} \left(\frac{2-1/n}{2+1/n}|x|\right) = |x|$. By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$ converges when $|x| < 1$, so $R = 1$. When $x = 1$, the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges by direct comparison with $\sum_{n=1}^{\infty} \frac{1}{2n}$ since $\frac{1}{2n-1} > \frac{1}{2n}$ and $\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges since it is a constant multiple of the harmonic series. When $x = -1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ converges by the Alternating Series Test. Thus, the interval of convergence is $[-1, 1)$.

12. If $a_n = \frac{(-1)^n x^n}{n^2}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-1)^n x^n}\right| = \lim_{n\to\infty} \left|\frac{(-1)xn^2}{(n+1)^2}\right| = \lim_{n\to\infty} \left[\left(\frac{n}{n+1}\right)^2 |x|\right] = 1^2 \cdot |x| = |x|.$$

[continued]

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$ converges when $|x| < 1$, so $R = 1$. When $x = 1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges by the Alternating Series Test. When $x = -1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges since it is a p-series with $p = 2 > 1$. Thus, the interval of convergence is $[-1, 1]$.

13. If $a_n = \frac{x^n}{n!}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = |x| \cdot 0 = 0 < 1$ for *all* real x. So, by the Ratio Test, $R = \infty$ and $I = (-\infty, \infty)$.

14. Here the Root Test is easier. If $a_n = n^n x^n$, then $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} n\,|x| = \infty$ if $x \neq 0$, so $R = 0$ and $I = \{0\}$.

15. If $a_n = \frac{x^n}{n^4\,4^n}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^4\,4^{n+1}} \cdot \frac{n^4\,4^n}{x^n}\right| = \lim_{n\to\infty} \left|\frac{n^4}{(n+1)^4} \cdot \frac{x}{4}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^4 \frac{|x|}{4} = 1^4 \cdot \frac{|x|}{4} = \frac{|x|}{4}.$$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{x^n}{n^4\,4^n}$ converges when $\frac{|x|}{4} < 1 \iff |x| < 4$, so $R = 4$. When $x = 4$, the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges since it is a p-series ($p = 4 > 1$). When $x = -4$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ converges by the Alternating Series Test. Thus, the interval of convergence is $[-4, 4]$.

16. If $a_n = 2^n n^2 x^n$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}(n+1)^2 x^{n+1}}{2^n n^2 x^n}\right| = \lim_{n\to\infty} 2\left(\frac{n+1}{n}\right)^2 |x| = 2\,|x|$. By the Ratio Test, the series $\sum_{n=1}^{\infty} 2^n n^2 x^n$ converges when $2\,|x| < 1 \iff |x| < \frac{1}{2}$, so $R = \frac{1}{2}$. When $x = \pm\frac{1}{2}$, both series $\sum_{n=1}^{\infty} 2^n n^2 \left(\pm\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} (\pm 1)^n n^2$ diverge by the Test for Divergence since $\lim_{n\to\infty} \left|(\pm 1)^n n^2\right| = \infty$. Thus, the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

17. If $a_n = \frac{(-1)^n\,4^n}{\sqrt{n}}\,x^n$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}\,4^{n+1}\,x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-1)^n\,4^n\,x^n}\right| = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} \cdot 4\,|x| = 4\,|x|$.

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n\,4^n}{\sqrt{n}}\,x^n$ converges when $4\,|x| < 1 \iff |x| < \frac{1}{4}$, so $R = \frac{1}{4}$. When $x = \frac{1}{4}$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. When $x = -\frac{1}{4}$, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges since it is a p-series $\left(p = \frac{1}{2} \leq 1\right)$. Thus, the interval of convergence is $\left(-\frac{1}{4}, \frac{1}{4}\right]$.

18. If $a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n}\,x^n$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^n x^{n+1}}{(n+1)\,5^{n+1}} \cdot \frac{n5^n}{(-1)^{n-1} x^n}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right) \frac{|x|}{5} = 1 \cdot \frac{|x|}{5} = \frac{|x|}{5}$.

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n}\,x^n$ converges when $\frac{|x|}{5} < 1 \iff |x| < 5$, so $R = 5$. When $x = 5$, the series

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$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges by the Alternating Series Test. When $x = -5$, the series $\sum_{n=1}^{\infty} \frac{-1}{n}$ diverges since it is a constant multiple of the harmonic series. Thus, the interval of convergence is $(-5, 5]$.

19. If $a_n = \frac{n}{2^n(n^2+1)} x^n$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{2^{n+1}(n^2+2n+2)} \cdot \frac{2^n(n^2+1)}{n\,x^n}\right| = \lim_{n\to\infty} \frac{n^3+n^2+n+1}{n^3+2n^2+2n} \cdot \frac{|x|}{2}$$

$$= \lim_{n\to\infty} \frac{1+1/n+1/n^2+1/n^3}{1+2/n+2/n^2} \cdot \frac{|x|}{2} = \frac{|x|}{2}$$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{n}{2^n(n^2+1)} x^n$ converges when $\frac{|x|}{2} < 1 \Leftrightarrow |x| < 2$, so $R = 2$. When $x = 2$, the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges by the Limit Comparison Test with $b_n = \frac{1}{n}$. When $x = -2$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ converges by the Alternating Series Test. Thus, the interval of convergence is $[-2, 2)$.

20. If $a_n = \frac{x^{2n}}{n!}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+2}}{(n+1)!} \cdot \frac{n!}{x^{2n}}\right| = \lim_{n\to\infty} \frac{|x^2|}{n+1} = 0 < 1$ for *all* real x. So, by the Ratio Test, $R = \infty$ and $I = (-\infty, \infty)$.

21. If $a_n = \frac{(x-2)^n}{n^2+1}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-2)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(x-2)^n}\right| = |x-2| \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1} = |x-2|$. By the Ratio Test, the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$ converges when $|x-2| < 1$ $[R = 1]$ $\Leftrightarrow$ $-1 < x-2 < 1$ $\Leftrightarrow$ $1 < x < 3$. When $x = 1$, the series $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n^2+1}$ converges by the Alternating Series Test; when $x = 3$, the series $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges by direct comparison with the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $[p = 2 > 1]$. Thus, the interval of convergence is $I = [1, 3]$.

22. If $a_n = \frac{(-1)^n}{(2n-1)2^n} (x-1)^n$, then

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(x-1)^{n+1}}{(2n+1)\,2^{n+1}} \cdot \frac{(2n-1)\,2^n}{(-1)^n(x-1)^n}\right| = \lim_{n\to\infty} \frac{2n-1}{2n+1} \cdot \frac{|x-1|}{2} = \frac{|x-1|}{2}$. By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)\,2^n} (x-1)^n$ converges when $\frac{|x-1|}{2} < 1$ $\Leftrightarrow$ $|x-1| < 2$ $[R = 2]$ $\Leftrightarrow$ $-2 < x-1 < 2$ $\Leftrightarrow$ $-1 < x < 3$. When $x = 3$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ converges by the Alternating Series Test. When $x = -1$, the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges by the Limit Comparison Test with $b_n = \frac{1}{n}$. Thus, the interval of convergence is $(-1, 3]$.

23. If $a_n = \frac{(x+2)^n}{2^n \ln n}$, then $\lim_{n\to\infty} \left|\frac{(x+2)^{n+1}}{2^{n+1}\ln(n+1)} \cdot \frac{2^n \ln n}{(x+2)^n}\right| = \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \cdot \frac{|x+2|}{2} = \frac{|x+2|}{2}$ since

$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} \overset{H}{=} \lim_{x\to\infty} \frac{1/x}{1/(x+1)} = \lim_{x\to\infty} \frac{x+1}{x} = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right) = 1$. By the Ratio Test, the series

$\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n \ln n}$ converges when $\frac{|x+2|}{2} < 1 \;\Leftrightarrow\; |x+2| < 2 \;\; [R = 2] \;\Leftrightarrow\; -2 < x+2 < 2 \;\Leftrightarrow\; -4 < x < 0.$

When $x = -4$, the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges by the Alternating Series Test. When $x = 0$, the series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by the Limit Comparison Test with $b_n = \frac{1}{n}$ (or by direct comparison with the harmonic series). Thus, the interval of convergence is $[-4, 0)$.

24. If $a_n = \frac{\sqrt{n}}{8^n}(x+6)^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}\,(x+6)^{n+1}}{8^{n+1}} \cdot \frac{8^n}{\sqrt{n}(x+6)^n}\right| = \lim_{n\to\infty}\sqrt{\frac{n+1}{n}} \cdot \frac{|x+6|}{8}$$
$$= \lim_{n\to\infty}\sqrt{1+\frac{1}{n}} \cdot \frac{|x+6|}{8} = \frac{|x+6|}{8}$$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n}(x+6)^n$ converges when $\frac{|x+6|}{8} < 1 \;\Leftrightarrow\; |x+6| < 8 \;\; [R = 8] \;\Leftrightarrow$ $-8 < x+6 < 8 \;\Leftrightarrow\; -14 < x < 2$. When $x = 2$, the series $\sum_{n=1}^{\infty} \sqrt{n}$ diverges by the Test for Divergence since $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \sqrt{n} = \infty > 0$. Similarly, when $x = -14$, the series $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$ diverges. Thus, the interval of convergence is $(-14, 2)$.

25. If $a_n = \frac{(x-2)^n}{n^n}$, then $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|x-2|}{n} = 0$, so the series converges for all x (by the Root Test). $R = \infty$ and $I = (-\infty, \infty)$.

26. If $a_n = \frac{(2x-1)^n}{5^n\sqrt{n}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}} \cdot \frac{5^n\sqrt{n}}{(2x-1)^n}\right| = \lim_{n\to\infty}\frac{|2x-1|}{5}\sqrt{\frac{n}{n+1}} = \lim_{n\to\infty}\frac{|2x-1|}{5}\sqrt{\frac{1}{1+1/n}} = \frac{|2x-1|}{5}.$$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$ converges when $\frac{|2x-1|}{5} < 1 \;\Leftrightarrow\; |2x-1| < 5 \;\Leftrightarrow\; \left|x - \frac{1}{2}\right| < \frac{5}{2} \;\Leftrightarrow$ $-\frac{5}{2} < x - \frac{1}{2} < \frac{5}{2} \;\Leftrightarrow\; -2 < x < 3$, so $R = \frac{5}{2}$. When $x = 3$, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series $\left(p = \frac{1}{2} \le 1\right)$. When $x = -2$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I = [-2, 3)$.

27. If $a_n = \frac{\ln n}{n} x^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\ln(n+1)\,x^{n+1}}{n+1} \cdot \frac{n}{(\ln n)\,x^n}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\,\frac{\ln(n+1)}{\ln n}\,x\right|$$
$$= 1 \cdot 1 \cdot |x| = |x|$$

since $\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \overset{H}{=} \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$. By the Ratio Test, the series $\sum_{n=4}^{\infty} \frac{\ln n}{n} x^n$

32. $a_n = \dfrac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} = \dfrac{n^2 x^n}{2^n n!} = \dfrac{n x^n}{2^n (n-1)!}$, so

$\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{(n+1)\,|x|^{n+1}}{2^{n+1} n!} \cdot \dfrac{2^n (n-1)!}{n\,|x|^n} = \lim\limits_{n\to\infty} \dfrac{n+1}{n^2} \dfrac{|x|}{2} = 0$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

33. If $a_n = \dfrac{(5x-4)^n}{n^3}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(5x-4)^{n+1}}{(n+1)^3} \cdot \frac{n^3}{(5x-4)^n}\right| = \lim_{n\to\infty} |5x-4| \left(\frac{n}{n+1}\right)^3 = \lim_{n\to\infty} |5x-4| \left(\frac{1}{1+1/n}\right)^3$$
$$= |5x-4| \cdot 1 = |5x-4|$$

By the Ratio Test, $\sum\limits_{n=1}^{\infty} \dfrac{(5x-4)^n}{n^3}$ converges when $|5x-4| < 1 \iff \left|x - \frac{4}{5}\right| < \frac{1}{5} \iff -\frac{1}{5} < x - \frac{4}{5} < \frac{1}{5} \iff$ $\frac{3}{5} < x < 1$, so $R = \frac{1}{5}$. When $x = 1$, the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$ is a convergent p-series ($p = 3 > 1$). When $x = \frac{3}{5}$, the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^3}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I = \left[\frac{3}{5}, 1\right]$.

34. If $a_n = \dfrac{x^{2n}}{n\,(\ln n)^2}$, then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{2n+2}}{(n+1)[\ln(n+1)]^2} \cdot \dfrac{n\,(\ln n)^2}{x^{2n}}\right| = \left|x^2\right| \lim\limits_{n\to\infty} \dfrac{n\,(\ln n)^2}{(n+1)[\ln(n+1)]^2} = x^2$.

By the Ratio Test, the series $\sum\limits_{n=2}^{\infty} \dfrac{x^{2n}}{n\,(\ln n)^2}$ converges when $x^2 < 1 \iff |x| < 1$, so $R = 1$. When $x = \pm 1$, $x^{2n} = 1$, the series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\,(\ln n)^2}$ converges by the Integral Test (see Exercise 11.3.31). Thus, the interval of convergence is $I = [-1, 1]$.

35. If $a_n = \dfrac{x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$, then

$\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{n+1}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)(2n+1)} \cdot \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{x^n}\right| = \lim\limits_{n\to\infty} \dfrac{|x|}{2n+1} = 0 < 1$. Thus, by the Ratio Test, the series $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$ converges for *all* real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

36. If $a_n = \dfrac{n!\, x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$, then

$\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(n+1)!\, x^{n+1}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)(2n+1)} \cdot \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!\, x^n}\right| = \lim\limits_{n\to\infty} \dfrac{(n+1)\,|x|}{2n+1} = \frac{1}{2}\,|x|$.

By the Ratio Test, the series $\sum\limits_{n=1}^{\infty} a_n$ converges when $\frac{1}{2}\,|x| < 1 \Rightarrow |x| < 2$, so $R = 2$. When $x = \pm 2$,

$|a_n| = \dfrac{n!\, 2^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} = \dfrac{[1 \cdot 2 \cdot 3 \cdot \cdots \cdot n]\, 2^n}{[1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)]} = \dfrac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} > 1$, so both endpoint series diverge by the Test for Divergence. Thus, the interval of convergence is $I = (-2, 2)$.