HW #6, SEC 11.10 SOLUTIONS

6.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\dfrac{1}{1+x}$	$\frac{1}{3}$
1	$-\dfrac{1}{(1+x)^2}$	$-\frac{1}{9}$
2	$\dfrac{2}{(1+x)^3}$	$\frac{2}{27}$
3	$-\dfrac{6}{(1+x)^4}$	$-\frac{6}{81}$

$$\sum_{n=0}^{3} \frac{f^{(n)}(2)}{n!}(x-2)^n = \frac{\frac{1}{3}}{0!}(x-2)^0 - \frac{\frac{1}{9}}{1!}(x-2)^1 + \frac{\frac{2}{27}}{2!}(x-2)^2 - \frac{\frac{6}{81}}{3!}(x-2)^3$$
$$= \tfrac{1}{3} - \tfrac{1}{9}(x-2) + \tfrac{1}{27}(x-2)^2 - \tfrac{1}{81}(x-2)^3$$

7.

n	$f^{(n)}(x)$	$f^{(n)}(8)$
0	$\sqrt[3]{x}$	2
1	$\dfrac{1}{3x^{2/3}}$	$\frac{1}{12}$
2	$-\dfrac{2}{9x^{5/3}}$	$-\frac{2}{288}$
3	$\dfrac{10}{27x^{8/3}}$	$\frac{10}{6912}$

$$\sum_{n=0}^{3} \frac{f^{(n)}(8)}{n!}(x-8)^n = \frac{2}{0!}(x-8)^0 + \frac{\frac{1}{12}}{1!}(x-8)^1 - \frac{\frac{2}{288}}{2!}(x-8)^2 + \frac{\frac{10}{6912}}{3!}(x-8)^3$$
$$= 2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2 + \tfrac{5}{20{,}736}(x-8)^3$$

8.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln x$	0
1	$1/x$	1
2	$-1/x^2$	-1
3	$2/x^3$	2
4	$-6/x^4$	-6

$$\sum_{n=0}^{4} \frac{f^{(n)}(1)}{n!}(x-1)^n = \frac{0}{0!}(x-1)^0 + \frac{1}{1!}(x-1)^1 - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{6}{4!}(x-1)^4$$
$$= (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4$$

9.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/6)$
0	$\sin x$	$1/2$
1	$\cos x$	$\sqrt{3}/2$
2	$-\sin x$	$-1/2$
3	$-\cos x$	$-\sqrt{3}/2$

$$\sum_{n=0}^{3} \frac{f^{(n)}(\pi/6)}{n!}\left(x-\frac{\pi}{6}\right)^n = \frac{1/2}{0!}\left(x-\frac{\pi}{6}\right)^0 + \frac{\sqrt{3}/2}{1!}\left(x-\frac{\pi}{6}\right)^1 - \frac{1/2}{2!}\left(x-\frac{\pi}{6}\right)^2 - \frac{\sqrt{3}/2}{3!}\left(x-\frac{\pi}{6}\right)^3$$
$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right) - \frac{1}{4}\left(x-\frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3$$

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10.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos^2 x$	1
1	$-2\cos x\,\sin x = -\sin 2x$	0
2	$-2\cos 2x$	-2
3	$4\sin 2x$	0
4	$8\cos 2x$	8
5	$-16\sin 2x$	0
6	$-32\cos 2x$	-32

$$\sum_{n=0}^{6} \frac{f^{(n)}(0)}{n!}(x-0)^n = \frac{1}{0!}x^0 - \frac{2}{2!}x^2 + \frac{8}{4!}x^4 - \frac{32}{6!}x^6$$
$$= 1 - x^2 + \tfrac{1}{3}x^4 - \tfrac{2}{45}x^6$$

11.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1-x)^{-2}$	1
1	$2(1-x)^{-3}$	2
2	$6(1-x)^{-4}$	6
3	$24(1-x)^{-5}$	24
4	$120(1-x)^{-6}$	120
$\vdots$	$\vdots$	$\vdots$

$$(1-x)^{-2} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$
$$= 1 + 2x + \tfrac{6}{2}x^2 + \tfrac{24}{6}x^3 + \tfrac{120}{24}x^4 + \cdots$$
$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \sum_{n=0}^{\infty}(n+1)x^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right| = |x|\lim_{n\to\infty}\frac{n+2}{n+1} = |x|\,(1) = |x| < 1$$

for convergence, so $R = 1$.

12.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$-(1+x)^{-2}$	-1
3	$2(1+x)^{-3}$	2
4	$-6(1+x)^{-4}$	-6
5	$24(1+x)^{-5}$	24
$\vdots$	$\vdots$	$\vdots$

$$\ln(1+x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$
$$+ \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots$$
$$= 0 + x - \tfrac{1}{2}x^2 + \tfrac{2}{6}x^3 - \tfrac{6}{24}x^4 + \tfrac{24}{120}x^5 - \cdots$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{1+1/n} = |x| < 1 \text{ for convergence,}$$

so $R = 1$.

Notice that the answer agrees with the entry for $\ln(1+x)$ in Table 1, but we obtained it by a different method. (Compare with Example 11.9.5.)

13.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1
$\vdots$	$\vdots$	$\vdots$

$$\cos x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$
$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \qquad \text{[Agrees with (16).]}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right| = \lim_{n\to\infty}\frac{x^2}{(2n+2)(2n+1)} = 0 < 1$$

for all x, so $R = \infty$.

28.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\cos x$	0
1	$-\sin x$	-1
2	$-\cos x$	0
3	$\sin x$	1
4	$\cos x$	0
5	$-\sin x$	-1
6	$-\cos x$	0
7	$\sin x$	1
$\vdots$	$\vdots$	$\vdots$

$$f(x) = \cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)}{n!}\left(x - \frac{\pi}{2}\right)^n$$

$$= \frac{-1}{1!}\left(x - \frac{\pi}{2}\right)^1 + \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3 + \frac{-1}{5!}\left(x - \frac{\pi}{2}\right)^5 + \frac{1}{7!}\left(x - \frac{\pi}{2}\right)^7 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}\left(x - \frac{\pi}{2}\right)^{2n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\left(x - \frac{\pi}{2}\right)^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^{n+1}\left(x - \frac{\pi}{2}\right)^{2n+1}}\right|$$

$$= \lim_{n\to\infty} \frac{\left(x - \frac{\pi}{2}\right)^2}{(2n+3)(2n+2)} = 0 < 1 \quad \text{for all } x, \text{ so } R = \infty.$$

29.

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\sin 2x$	0
1	$2\cos 2x$	2
2	$-4\sin 2x$	0
3	$-8\cos 2x$	-8
4	$16\sin 2x$	0
5	$32\cos 2x$	32
$\vdots$	$\vdots$	$\vdots$

$$f(x) = \sin 2x = f(\pi) + \frac{f'(\pi)}{1!}(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2$$

$$+ \frac{f'''(\pi)}{3!}(x-\pi)^3 + \frac{f^{(4)}(\pi)}{4!}(x-\pi)^4 + \cdots$$

$$= 0 + \frac{2}{1!}(x-\pi) + 0 - \frac{8}{3!}(x-\pi)^3 + 0 + \frac{32}{5!}(x-\pi)^5 - \cdots$$

$$= \frac{2}{1!}(x-\pi) - \frac{8}{3!}(x-\pi)^3 + \frac{32}{5!}(x-\pi)^5 - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1}}{(2n+1)!}(x-\pi)^{2n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\, 2^{2n+3}\,(x-\pi)^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n\, 2^{2n+1}\,(x-\pi)^{2n+1}}\right|$$

$$= (x-\pi)^2 \lim_{n\to\infty} \frac{2^2}{(2n+3)(2n+2)} = 0 < 1 \text{ for all } x, \text{ so } R = \infty.$$

30.

n	$f^{(n)}(x)$	$f^{(n)}(16)$
0	$\sqrt{x}$	4
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}\cdot\frac{1}{4}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{4}\cdot\frac{1}{4^3}$
3	$\frac{3}{8}x^{-5/2}$	$\frac{3}{8}\cdot\frac{1}{4^5}$
4	$-\frac{15}{16}x^{-7/2}$	$-\frac{15}{16}\cdot\frac{1}{4^7}$
$\vdots$	$\vdots$	$\vdots$

$$f(x) = \sqrt{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(16)}{n!}(x-16)^n$$

$$= \frac{4}{0!}(x-16)^0 + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{1!}(x-16)^1 - \frac{1}{4}\cdot\frac{1}{4^3}\cdot\frac{1}{2!}(x-16)^2$$

$$+ \frac{3}{8}\cdot\frac{1}{4^5}\cdot\frac{1}{3!}(x-16)^3 - \frac{15}{16}\cdot\frac{1}{4^7}\cdot\frac{1}{4!}(x-16)^4 + \cdots$$

$$= 4 + \frac{1}{8}(x-16) + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n 4^{2n-1}\, n!}(x-16)^n$$

$$= 4 + \frac{1}{8}(x-16) + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^{5n-2}\, n!}(x-16)^n$$

[continued]

36. $\sqrt[3]{8+x} = \sqrt[3]{8\left(1+\frac{x}{8}\right)} = 2\left(1+\frac{x}{8}\right)^{1/3} = 2\sum_{n=0}^{\infty}\binom{1/3}{n}\left(\frac{x}{8}\right)^n$

$$= 2\left[1+\frac{1}{3}\left(\frac{x}{8}\right)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}\left(\frac{x}{8}\right)^2+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{x}{8}\right)^3+\cdots\right]$$

$$= 2\left[1+\frac{1}{24}x+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cdot[2\cdot 5\cdot\cdots\cdot(3n-4)]}{3^n\cdot 8^n\cdot n!}x^n\right]$$

$$= 2+\frac{1}{12}x+2\sum_{n=2}^{\infty}\frac{(-1)^{n-1}[2\cdot 5\cdot\cdots\cdot(3n-4)]}{24^n\cdot n!}x^n$$

and $\left|\frac{x}{8}\right| < 1 \quad\Leftrightarrow\quad |x| < 8$, so $R = 8$.

37. $\frac{1}{(2+x)^3} = \frac{1}{[2(1+x/2)]^3} = \frac{1}{8}\left(1+\frac{x}{2}\right)^{-3} = \frac{1}{8}\sum_{n=0}^{\infty}\binom{-3}{n}\left(\frac{x}{2}\right)^n$. The binomial coefficient is

$$\binom{-3}{n} = \frac{(-3)(-4)(-5)\cdot\cdots\cdot(-3-n+1)}{n!} = \frac{(-3)(-4)(-5)\cdot\cdots\cdot[-(n+2)]}{n!}$$

$$= \frac{(-1)^n\cdot 2\cdot 3\cdot 4\cdot 5\cdot\cdots\cdot(n+1)(n+2)}{2\cdot n!} = \frac{(-1)^n(n+1)(n+2)}{2}$$

Thus, $\frac{1}{(2+x)^3} = \frac{1}{8}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)}{2}\frac{x^n}{2^n} = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)x^n}{2^{n+4}}$ for $\left|\frac{x}{2}\right| < 1 \quad\Leftrightarrow\quad |x| < 2$, so $R = 2$.

38. $(1-x)^{3/4} = [1+(-x)]^{3/4} = \sum_{n=0}^{\infty}\binom{3/4}{n}(-x)^n = 1+\frac{3}{4}(-x)+\frac{\frac{3}{4}\left(-\frac{1}{4}\right)}{2!}(-x)^2+\frac{\frac{3}{4}\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{3!}(-x)^3+\cdots$

$$= 1-\frac{3}{4}x+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(-1)^n\cdot 3\cdot[1\cdot 5\cdot 9\cdot\cdots\cdot(4n-7)]}{4^n\cdot n!}x^n$$

$$= 1-\frac{3}{4}x-3\sum_{n=2}^{\infty}\frac{1\cdot 5\cdot 9\cdot\cdots\cdot(4n-7)}{4^n\cdot n!}x^n$$

and $|-x| < 1 \quad\Leftrightarrow\quad |x| < 1$, so $R = 1$.

39. $\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$, so $f(x) = \arctan(x^2) = \sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}x^{4n+2}$, $R = 1$.

40. $\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$, so $f(x) = \sin\left(\frac{\pi}{4}x\right) = \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{\pi}{4}x\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!}x^{2n+1}$, $R = \infty$.

41. $\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \quad\Rightarrow\quad \cos 2x = \sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}x^{2n}}{(2n)!}$, so

$f(x) = x\cos 2x = \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}}{(2n)!}x^{2n+1}$, $R = \infty$.

42. $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, so $f(x) = e^{3x}-e^{2x} = \sum_{n=0}^{\infty}\frac{(3x)^n}{n!}-\sum_{n=0}^{\infty}\frac{(2x)^n}{n!} = \sum_{n=0}^{\infty}\frac{3^nx^n}{n!}-\sum_{n=0}^{\infty}\frac{2^nx^n}{n!} = \sum_{n=0}^{\infty}\frac{3^n-2^n}{n!}x^n$, $R = \infty$.

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43. $\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} \quad \Rightarrow \quad \cos\left(\frac{1}{2}x^2\right) = \sum_{n=0}^{\infty} (-1)^n \dfrac{\left(\frac{1}{2}x^2\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{4n}}{2^{2n}(2n)!}$, so

$$f(x) = x\cos\left(\tfrac{1}{2}x^2\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n}(2n)!} x^{4n+1}, \ R = \infty.$$

44. $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n} \quad \Rightarrow \quad \ln(1+x^3) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^{3n}}{n}$, so $f(x) = x^2 \ln(1+x^3) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^{3n+2}}{n}$,

$R = 1$.

45. We must write the binomial in the form (1+ expression), so we'll factor out a 4.

$$\frac{x}{\sqrt{4+x^2}} = \frac{x}{\sqrt{4(1+x^2/4)}} = \frac{x}{2\sqrt{1+x^2/4}} = \frac{x}{2}\left(1+\frac{x^2}{4}\right)^{-1/2} = \frac{x}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x^2}{4}\right)^n$$

$$= \frac{x}{2}\left[1 + \left(-\tfrac{1}{2}\right)\frac{x^2}{4} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x^2}{4}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x^2}{4}\right)^3 + \cdots\right]$$

$$= \frac{x}{2} + \frac{x}{2}\sum_{n=1}^{\infty}(-1)^n \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n\cdot 4^n\cdot n!}x^{2n}$$

$$= \frac{x}{2} + \sum_{n=1}^{\infty}(-1)^n \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{3n+1}}x^{2n+1} \text{ and } \frac{x^2}{4} < 1 \ \Leftrightarrow \ \frac{|x|}{2} < 1 \ \Leftrightarrow \ |x| < 2, \text{ so } R = 2.$$

46. $$\frac{x^2}{\sqrt{2+x}} = \frac{x^2}{\sqrt{2(1+x/2)}} = \frac{x^2}{\sqrt{2}}\left(1+\frac{x}{2}\right)^{-1/2} = \frac{x^2}{\sqrt{2}}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x}{2}\right)^n$$

$$= \frac{x^2}{\sqrt{2}}\left[1 + \left(-\tfrac{1}{2}\right)\left(\frac{x}{2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{2}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x}{2}\right)^3 + \cdots\right]$$

$$= \frac{x^2}{\sqrt{2}} + \frac{x^2}{\sqrt{2}}\sum_{n=1}^{\infty}(-1)^n \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{2n}}x^n$$

$$= \frac{x^2}{\sqrt{2}} + \sum_{n=1}^{\infty}(-1)^n \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{2n+1/2}}x^{n+2} \text{ and } \left|\frac{x}{2}\right| < 1 \ \Leftrightarrow \ |x| < 2, \text{ so } R = 2.$$

47. $\sin^2 x = \dfrac{1}{2}(1-\cos 2x) = \dfrac{1}{2}\left[1 - \sum_{n=0}^{\infty}\dfrac{(-1)^n(2x)^{2n}}{(2n)!}\right] = \dfrac{1}{2}\left[1 - 1 - \sum_{n=1}^{\infty}\dfrac{(-1)^n(2x)^{2n}}{(2n)!}\right] = \sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}2^{2n-1}x^{2n}}{(2n)!}$,

$R = \infty$

48. $$\frac{x-\sin x}{x^3} = \frac{1}{x^3}\left[x - \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] = \frac{1}{x^3}\left[x - x - \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] = \frac{1}{x^3}\left[-\sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}\right]$$

$$= \frac{1}{x^3}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{(2n+3)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+3)!}$$

and this series also gives the required value at $x = 0$ (namely $1/6$); $R = \infty$.

56. $1/\sqrt[10]{e} = e^{-1/10}$ and $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, so

$e^{-1/10} = 1 - \frac{1}{10} + \frac{(1/10)^2}{2!} - \frac{(1/10)^3}{3!} + \frac{(1/10)^4}{4!} - \frac{(1/10)^5}{5!} + \cdots$. Now

$1 - \frac{1}{10} + \frac{(1/10)^2}{2!} - \frac{(1/10)^3}{3!} + \frac{(1/10)^4}{4!} \approx 0.90484$ and subtracting $\frac{(1/10)^5}{5!} \approx 8.3 \times 10^{-8}$ does not affect the fifth decimal place, so $e^{-1/10} \approx 0.90484$ by the Alternating Series Estimation Theorem.

57. (a) $1/\sqrt{1-x^2} = \left[1 + (-x^2)\right]^{-1/2} = 1 + \left(-\frac{1}{2}\right)(-x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \cdots$

$$= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n \cdot n!} x^{2n}$$

(b) $\sin^{-1} x = \int \frac{1}{\sqrt{1-x^2}}\, dx = C + x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n+1)2^n \cdot n!} x^{2n+1}$

$$= x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n+1)2^n \cdot n!} x^{2n+1} \quad \text{since } 0 = \sin^{-1} 0 = C.$$

58. (a) $1/\sqrt[4]{1+x} = (1+x)^{-1/4} = \sum_{n=0}^{\infty} \binom{-\frac{1}{4}}{n} x^n = 1 - \frac{1}{4}x + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!}x^2 + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{3!}x^3 + \cdots$

$$= 1 - \frac{1}{4}x + \sum_{n=2}^{\infty} (-1)^n \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{4^n \cdot n!} x^n$$

(b) $1/\sqrt[4]{1+x} = 1 - \frac{1}{4}x + \frac{5}{32}x^2 - \frac{15}{128}x^3 + \frac{195}{2048}x^4 - \cdots$. $1/\sqrt[4]{1.1} = 1/\sqrt[4]{1+0.1}$, so let $x = 0.1$. The sum of the first four terms is then $1 - \frac{1}{4}(0.1) + \frac{5}{32}(0.1)^2 - \frac{15}{128}(0.1)^3 \approx 0.976$. The fifth term is $\frac{195}{2048}(0.1)^4 \approx 0.000\,009\,5$, which does not affect the third decimal place of the sum, so we have $1/\sqrt[4]{1.1} \approx 0.976$. (Note that the third decimal place of the sum of the first three terms is affected by the fourth term, so we need to use more than three terms for the sum.)

59. $\sqrt{1+x^3} = (1+x^3)^{1/2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (x^3)^n = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^{3n} \Rightarrow \int \sqrt{1+x^3}\, dx = C + \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{x^{3n+1}}{3n+1}$,

with $R = 1$.

60. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \Rightarrow \sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} \Rightarrow$

$x^2 \sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+4}}{(2n+1)!} \Rightarrow \int x^2 \sin(x^2)\, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+5}}{(2n+1)!(4n+5)}$, with $R = \infty$.

61. $\cos x \overset{(16)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow \cos x - 1 = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow \frac{\cos x - 1}{x} = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n)!} \Rightarrow$

$\int \frac{\cos x - 1}{x}\, dx = C + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{2n \cdot (2n)!}$, with $R = \infty$.

81. $\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2^{2n+1}(2n+1)} = \sum_{n=0}^{\infty}(-1)^n\dfrac{(x/2)^{2n+1}}{2n+1} = \tan^{-1}\left(\dfrac{x}{2}\right)$ [from Table 1]

82. $\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\dfrac{(x/2)^{2n+1}}{(2n+1)!} = \sin\dfrac{x}{2}$ by (15).

83. $\sum_{n=0}^{\infty}\dfrac{(-1)^n}{n!}$ is the Maclaurin series for e^x evaluated at $x=-1$. Thus, $\sum_{n=0}^{\infty}\dfrac{(-1)^n}{n!} = e^{-1}$ by (11).

84. $\sum_{n=0}^{\infty}\dfrac{(-1)^n\pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty}(-1)^n\dfrac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, by (16).

85. $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{3^n}{n5^n} = \sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{(3/5)^n}{n} = \ln\left(1+\dfrac{3}{5}\right)$ [from Table 1] $= \ln\dfrac{8}{5}$

86. $\sum_{n=0}^{\infty}\dfrac{3^n}{5^n\,n!} = \sum_{n=0}^{\infty}\dfrac{(3/5)^n}{n!} = e^{3/5}$, by (11).

87. $\sum_{n=0}^{\infty}\dfrac{(-1)^n\pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}\dfrac{(-1)^n\left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, by (15).

88. $1-\ln 2+\dfrac{(\ln 2)^2}{2!}-\dfrac{(\ln 2)^3}{3!}+\cdots = \sum_{n=0}^{\infty}\dfrac{(-\ln 2)^n}{n!} = e^{-\ln 2} = \left(e^{\ln 2}\right)^{-1} = 2^{-1} = \frac{1}{2}$, by (11).

89. $3+\dfrac{9}{2!}+\dfrac{27}{3!}+\dfrac{81}{4!}+\cdots = \dfrac{3^1}{1!}+\dfrac{3^2}{2!}+\dfrac{3^3}{3!}+\dfrac{3^4}{4!}+\cdots = \sum_{n=1}^{\infty}\dfrac{3^n}{n!} = \sum_{n=0}^{\infty}\dfrac{3^n}{n!}-1 = e^3-1$, by (11).

90. $$\frac{1}{1\cdot 2}-\frac{1}{3\cdot 2^3}+\frac{1}{5\cdot 2^5}-\frac{1}{7\cdot 2^7}+\cdots = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)2^{2n+1}}$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{(1/2)^{2n+1}}{2n+1} = \tan^{-1}\left(\frac{1}{2}\right) \quad \text{[from Table 1]}$$

91. If p is an nth-degree polynomial, then $p^{(i)}(x)=0$ for $i>n$, so its Taylor series at a is $p(x) = \sum_{i=0}^{n}\dfrac{p^{(i)}(a)}{i!}(x-a)^i$.

Put $x-a=1$, so that $x=a+1$. Then $p(a+1) = \sum_{i=0}^{n}\dfrac{p^{(i)}(a)}{i!}$.

This is true for any a, so replace a by x: $p(x+1) = \sum_{i=0}^{n}\dfrac{p^{(i)}(x)}{i!}$

92. Using the geometric series from Table 1, we have

$$\frac{x}{1+x^2} = x\cdot\frac{1}{1-(-x^2)} = x\cdot\sum_{n=0}^{\infty}(-x^2)^n = x\cdot\sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}.$$

The x^{101} term is obtained when $n=50$ and is $(-1)^{50}x^{101} = x^{101}$. So the coefficient of x^{101} is 1 which, by Equation 7, must equal $\dfrac{f^{(101)}(0)}{101!}$. Thus, $\dfrac{f^{(101)}(0)}{101!} = 1 \;\Rightarrow\; f^{(101)}(0) = 101!$.