HW #7, PART VI - A ; SECTION 10.5 Solutions
(ON PARABOLAS)

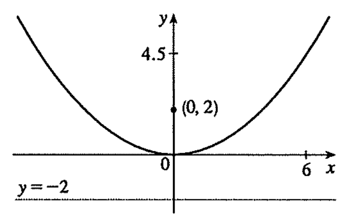
$ds = \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta = \sqrt{r^2 + (dr/d\theta)^2}\,d\theta$ [see the derivation of Equation 10.4.6]. Therefore, for a polar equation rotated around $\theta = \frac{\pi}{2}$, $S = \int_a^b 2\pi r \cos\theta \sqrt{r^2 + (dr/d\theta)^2}\,d\theta$.

(b) As in the solution for Exercise 75(b), we can double the surface area generated by rotating the curve from $\theta = 0$ to $\theta = \frac{\pi}{4}$ to obtain the total surface area.

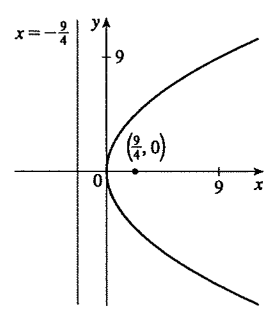
$$\begin{aligned} S &= 2\int_0^{\pi/4} 2\pi \sqrt{\cos 2\theta}\cos\theta \sqrt{\cos 2\theta + (\sin^2 2\theta)/\cos 2\theta}\,d\theta \\ &= 4\pi \int_0^{\pi/4} \sqrt{\cos 2\theta}\cos\theta \sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}}\,d\theta \\ &= 4\pi \int_0^{\pi/4} \sqrt{\cos 2\theta}\cos\theta \frac{1}{\sqrt{\cos 2\theta}}\,d\theta = 4\pi \int_0^{\pi/4} \cos\theta\,d\theta \\ &= 4\pi\big[\sin\theta\big]_0^{\pi/4} = 4\pi\left(\frac{\sqrt{2}}{2} - 0\right) = 2\sqrt{2}\,\pi \end{aligned}$$

10.5 Conic Sections

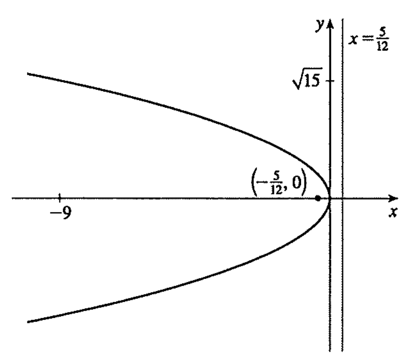
1. $x^2 = 8y$ and $x^2 = 4py \Rightarrow 4p = 8 \Leftrightarrow p = 2$. The vertex is $(0, 0)$, the focus is $(0, 2)$, and the directrix is $y = -2$.



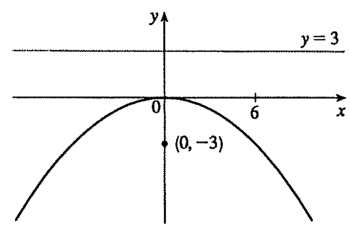
2. $9x = y^2$, so $4p = 9 \Leftrightarrow p = \frac{9}{4}$. The vertex is $(0, 0)$, the focus is $\left(\frac{9}{4}, 0\right)$, and the directrix is $x = -\frac{9}{4}$.



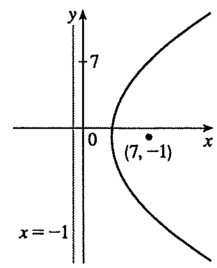
3. $5x + 3y^2 = 0 \Leftrightarrow y^2 = -\frac{5}{3}x$, so $4p = -\frac{5}{3} \Leftrightarrow p = -\frac{5}{12}$. The vertex is $(0, 0)$, the focus is $\left(-\frac{5}{12}, 0\right)$, and the directrix is $x = \frac{5}{12}$.



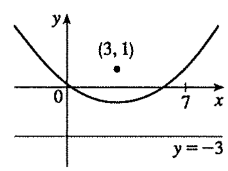
4. $x^2 + 12y = 0 \Leftrightarrow x^2 = -12y$, so $4p = -12 \Leftrightarrow p = -3$. The vertex is $(0, 0)$, the focus is $(0, -3)$, and the directrix is $y = 3$.



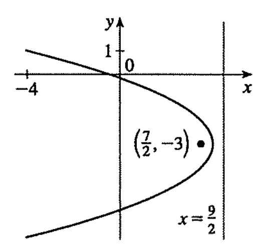
5. $(y + 1)^2 = 16(x - 3)$, so $4p = 16 \Leftrightarrow p = 4$. The vertex is $(3, -1)$, the focus is $(3 + 4, -1) = (7, -1)$, and the directrix is $x = 3 - 4 = -1$.



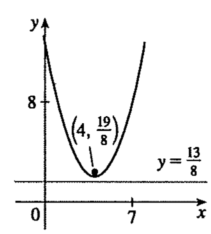
6. $(x - 3)^2 = 8(y + 1)$, so $4p = 8 \Leftrightarrow p = 2$. The vertex is $(3, -1)$, the focus is $(3, -1 + 2) = (3, 1)$, and the directrix is $y = -1 - 2 = -3$.



7. $y^2 + 6y + 2x + 1 = 0 \Leftrightarrow y^2 + 6y = -2x - 1 \Leftrightarrow$ $y^2 + 6y + 9 = -2x + 8 \Leftrightarrow (y + 3)^2 = -2(x - 4)$, so $4p = -2 \Leftrightarrow$ $p = -\frac{1}{2}$. The vertex is $(4, -3)$, the focus is $\left(\frac{7}{2}, -3\right)$, and the directrix is $x = \frac{9}{2}$.



8. $2x^2 - 16x - 3y + 38 = 0 \Leftrightarrow 2x^2 - 16x = 3y - 38 \Leftrightarrow$ $2(x^2 - 8x + 16) = 3y - 38 + 32 \Leftrightarrow 2(x - 4)^2 = 3y - 6 \Leftrightarrow$ $(x - 4)^2 = \frac{3}{2}(y - 2)$, so $4p = \frac{3}{2} \Leftrightarrow p = \frac{3}{8}$. The vertex is $(4, 2)$, the focus is $\left(4, \frac{19}{8}\right)$, and the directrix is $y = \frac{13}{8}$.



9. The equation has the form $y^2 = 4px$, where $p < 0$. Since the parabola passes through $(-1, 1)$, we have $1^2 = 4p(-1)$, so $4p = -1$ and an equation is $y^2 = -x$ or $x = -y^2$. $4p = -1$, so $p = -\frac{1}{4}$ and the focus is $\left(-\frac{1}{4}, 0\right)$ while the directrix is $x = \frac{1}{4}$.

10. The vertex is $(2, -2)$, so the equation is of the form $(x - 2)^2 = 4p(y + 2)$, where $p > 0$. The point $(0, 0)$ is on the parabola, so $4 = 4p(2) \Rightarrow 4p = 2$. Thus, an equation is $(x - 2)^2 = 2(y + 2)$. $4p = 2$, so $p = \frac{1}{2}$ and the focus is $\left(2, -\frac{3}{2}\right)$ while the directrix is $y = -\frac{5}{2}$.

28. $4x^2 = y + 4 \Leftrightarrow x^2 = \frac{1}{4}(y + 4)$. This is an equation of a *parabola* with $4p = \frac{1}{4}$, so $p = \frac{1}{16}$. The vertex is $(0, -4)$ and the focus is $\left(0, -4 + \frac{1}{16}\right) = \left(0, -\frac{63}{16}\right)$.

29. $x^2 = 4y - 2y^2 \Leftrightarrow x^2 + 2y^2 - 4y = 0 \Leftrightarrow x^2 + 2(y^2 - 2y + 1) = 2 \Leftrightarrow x^2 + 2(y - 1)^2 = 2 \Leftrightarrow$ $\frac{x^2}{2} + \frac{(y-1)^2}{1} = 1$. This is an equation of an *ellipse* with vertices at $\left(\pm\sqrt{2}, 1\right)$. The foci are at $\left(\pm\sqrt{2-1}, 1\right) = (\pm 1, 1)$.

30. $y^2 - 2 = x^2 - 2x \Leftrightarrow y^2 - x^2 + 2x = 2 \Leftrightarrow y^2 - (x^2 - 2x + 1) = 2 - 1 \Leftrightarrow \frac{y^2}{1} - \frac{(x-1)^2}{1} = 1$. This is an equation of a *hyperbola* with vertices $(1, \pm 1)$. The foci are at $\left(1, \pm\sqrt{1+1}\right) = \left(1, \pm\sqrt{2}\right)$.

31. $3x^2 - 6x - 2y = 1 \Leftrightarrow 3x^2 - 6x = 2y + 1 \Leftrightarrow 3(x^2 - 2x + 1) = 2y + 1 + 3 \Leftrightarrow 3(x-1)^2 = 2y + 4 \Leftrightarrow$ $(x-1)^2 = \frac{2}{3}(y + 2)$. This is an equation of a *parabola* with $4p = \frac{2}{3}$, so $p = \frac{1}{6}$. The vertex is $(1, -2)$ and the focus is $\left(1, -2 + \frac{1}{6}\right) = \left(1, -\frac{11}{6}\right)$.

32. $x^2 - 2x + 2y^2 - 8y + 7 = 0 \Leftrightarrow (x^2 - 2x + 1) + 2(y^2 - 4y + 4) = -7 + 1 + 8 \Leftrightarrow (x-1)^2 + 2(y-2)^2 = 2 \Leftrightarrow$ $\frac{(x-1)^2}{2} + \frac{(y-2)^2}{1} = 1$. This is an equation of an *ellipse* with vertices at $\left(1 \pm \sqrt{2}, 2\right)$. The foci are at $\left(1 \pm \sqrt{2-1}, 2\right) = (1 \pm 1, 2)$.

33. The parabola with vertex $(0, 0)$ and focus $(1, 0)$ opens to the right and has $p = 1$, so its equation is $y^2 = 4px$, or $y^2 = 4x$.

34. The parabola with focus $(0, 0)$ and directrix $y = 6$ has vertex $(0, 3)$ and opens downward, so $p = -3$ and its equation is $(x - 0)^2 = 4p(y - 3)$, or $x^2 = -12(y - 3)$.

35. The distance from the focus $(-4, 0)$ to the directrix $x = 2$ is $2 - (-4) = 6$, so the distance from the focus to the vertex is $\frac{1}{2}(6) = 3$ and the vertex is $(-1, 0)$. Since the focus is to the left of the vertex, $p = -3$. An equation is $y^2 = 4p(x + 1) \Rightarrow$ $y^2 = -12(x + 1)$.

36. The parabola with vertex $(2, 3)$ and focus $(2, -1)$ opens downward and has $p = -1 - 3 = -4$, so its equation is $(x-2)^2 = 4p(y - 3)$, or $(x-2)^2 = -16(y - 3)$.

37. The parabola with vertex $(3, -1)$ having a horizontal axis has equation $[y - (-1)]^2 = 4p(x - 3)$. Since it passes through $(-15, 2)$, $(2+1)^2 = 4p(-15 - 3) \Rightarrow 9 = 4p(-18) \Rightarrow 4p = -\frac{1}{2}$. An equation is $(y+1)^2 = -\frac{1}{2}(x - 3)$.

38. The parabola with vertical axis and passing through $(0, 4)$ has equation $y = ax^2 + bx + 4$. It also passes through $(1, 3)$ and $(-2, -6)$, so

$$\begin{cases} 3 = a + b + 4 \\ -6 = 4a - 2b + 4 \end{cases} \Rightarrow \begin{cases} -1 = a + b \\ -10 = 4a - 2b \end{cases} \Rightarrow \begin{cases} -1 = a + b \\ -5 = 2a - b \end{cases}$$

Adding the last two equations gives us $3a = -6$, or $a = -2$. Since $a + b = -1$, we have $b = 1$, and an equation is $y = -2x^2 + x + 4$.