

HW #7 SECTION 10.2 SOLUTIONS

10.2 Calculus with Parametric Curves

1. $x = 2t^3 + 3t$, $y = 4t - 5t^2 \Rightarrow \frac{dx}{dt} = 6t^2 + 3$, $\frac{dy}{dt} = 4 - 10t$, and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4 - 10t}{6t^2 + 3}$.

2. $x = t - \ln t$, $y = t^2 - t^{-2} \Rightarrow \frac{dx}{dt} = 1 - t^{-1}$, $\frac{dy}{dt} = 2t + 2t^{-3}$, and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t + 2t^{-3}}{1 - t^{-1}} \cdot \frac{t^3}{t^3} = \frac{2t^4 + 2}{t^3 - t^2}$.

3. $x = te^t$, $y = t + \sin t \Rightarrow \frac{dx}{dt} = te^t + e^t = e^t(t + 1)$, $\frac{dy}{dt} = 1 + \cos t$, and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 + \cos t}{e^t(t + 1)}$.

4. $x = t + \sin(t^2 + 2)$, $y = \tan(t^2 + 2) \Rightarrow \frac{dx}{dt} = 1 + 2t \cos(t^2 + 2)$, $\frac{dy}{dt} = 2t \sec^2(t^2 + 2)$, and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t \sec^2(t^2 + 2)}{1 + 2t \cos(t^2 + 2)}$.

5. $x = t^2 + 2t$, $y = 2^t - 2t$; (15, 2). $\frac{dy}{dt} = 2^t \ln 2 - 2$, $\frac{dx}{dt} = 2t + 2$, and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2^t \ln 2 - 2}{2t + 2}$.

At (15, 2), $x = t^2 + 2t = 15 \Rightarrow t^2 + 2t - 15 = 0 \Rightarrow (t + 5)(t - 3) = 0 \Rightarrow t = -5$ or $t = 3$. Only $t = 3$ gives

$y = 2$. With $t = 3$, $\frac{dy}{dx} = \frac{2^3 \ln 2 - 2}{2(3) + 2} = \frac{4 \ln 2 - 1}{4} = \ln 2 - \frac{1}{4} \approx 0.44$.

6. $x = t + \cos \pi t$, $y = -t + \sin \pi t$; (3, -2). $\frac{dy}{dt} = -1 + \pi \cos \pi t$, $\frac{dx}{dt} = 1 - \pi \sin \pi t$, and

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1 + \pi \cos \pi t}{1 - \pi \sin \pi t}$. When $x = 3$, we have $t + \cos \pi t = 3 \Rightarrow \cos^2 \pi t = (3 - t)^2$ (1). When $y = -2$, we

have $-t + \sin \pi t = -2 \Rightarrow \sin^2 \pi t = (t - 2)^2$ (2). Adding (1) and (2) gives $\sin^2 \pi t + \cos^2 \pi t = (t - 2)^2 + (3 - t)^2 \Rightarrow$

$1 = t^2 - 4t + 4 + 9 - 6t + t^2 \Rightarrow 0 = 2t^2 - 10t + 12 \Rightarrow 0 = 2(t - 2)(t - 3) \Rightarrow t = 2$ or $t = 3$. Only $t = 2$ gives

$y = -2$. With $t = 2$, $\frac{dy}{dx} = \frac{-1 + \pi \cos 2\pi}{1 - \pi \sin 2\pi} = \frac{-1 + \pi}{1 - 0} = \pi - 1 \approx 2.14$.

7. $x = t^3 + 1$, $y = t^4 + t$; $t = -1$. $\frac{dy}{dt} = 4t^3 + 1$, $\frac{dx}{dt} = 3t^2$, and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3 + 1}{3t^2}$. When $t = -1$, $(x, y) = (0, 0)$

and $dy/dx = -3/3 = -1$, so an equation of the tangent to the curve at the point corresponding to $t = -1$ is

$y - 0 = -1(x - 0)$, or $y = -x$.

8. $x = \sqrt{t}$, $y = t^2 - 2t$; $t = 4$. $\frac{dy}{dt} = 2t - 2$, $\frac{dx}{dt} = \frac{1}{2\sqrt{t}}$, and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = (2t - 2)2\sqrt{t} = 4(t - 1)\sqrt{t}$. When $t = 4$,

$(x, y) = (2, 8)$ and $dy/dx = 4(3)(2) = 24$, so an equation of the tangent to the curve at the point corresponding to $t = 4$ is

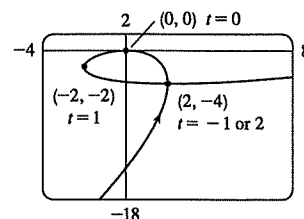
$y - 8 = 24(x - 2)$, or $y = 24x - 40$.

9. $x = \sin 2t + \cos t$, $y = \cos 2t - \sin t$; $t = \pi$. $\frac{dy}{dt} = -2 \sin 2t - \cos t$, $\frac{dx}{dt} = 2 \cos 2t - \sin t$, and

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2 \sin 2t - \cos t}{2 \cos 2t - \sin t}$. When $t = \pi$, $(x, y) = (-1, 1)$, and $\frac{dy}{dx} = \frac{1}{2}$, so an equation of the tangent to the curve at

the point corresponding to $t = \pi$ is $y - 1 = \frac{1}{2}[x - (-1)]$, or $y = \frac{1}{2}x + \frac{3}{2}$.

22. $x = t^3 - 3t$, $y = t^3 - 3t^2$. $\frac{dy}{dt} = 3t^2 - 6t = 3t(t - 2)$, so $\frac{dy}{dt} = 0 \Leftrightarrow$
 $t = 0$ or $2 \Leftrightarrow (x, y) = (0, 0)$ or $(2, -4)$. $\frac{dx}{dt} = 3t^2 - 3 = 3(t + 1)(t - 1)$,
 so $\frac{dx}{dt} = 0 \Leftrightarrow t = -1$ or $1 \Leftrightarrow (x, y) = (2, -4)$ or $(-2, -2)$. The curve
 has horizontal tangents at $(0, 0)$ and $(2, -4)$, and vertical tangents at $(2, -4)$
 and $(-2, -2)$.



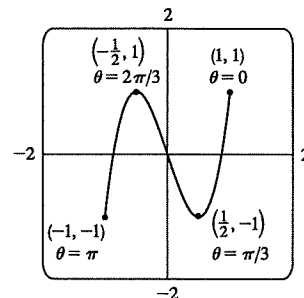
23. $x = \cos \theta$, $y = \cos 3\theta$. The whole curve is traced out for $0 \leq \theta \leq \pi$.

$$\frac{dy}{d\theta} = -3 \sin 3\theta, \text{ so } \frac{dy}{d\theta} = 0 \Leftrightarrow \sin 3\theta = 0 \Leftrightarrow 3\theta = 0, \pi, 2\pi, \text{ or } 3\pi \Leftrightarrow$$

$$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \text{ or } \pi \Leftrightarrow (x, y) = (1, 1), \left(\frac{1}{2}, -1\right), \left(-\frac{1}{2}, 1\right), \text{ or } (-1, -1).$$

$$\frac{dx}{d\theta} = -\sin \theta, \text{ so } \frac{dx}{d\theta} = 0 \Leftrightarrow \sin \theta = 0 \Leftrightarrow \theta = 0 \text{ or } \pi \Leftrightarrow$$

$$(x, y) = (1, 1) \text{ or } (-1, -1). \text{ Both } \frac{dy}{d\theta} \text{ and } \frac{dx}{d\theta} \text{ equal 0 when } \theta = 0 \text{ and } \pi.$$



To find the slope when $\theta = 0$, we find $\lim_{\theta \rightarrow 0} \frac{dy}{dx} = \lim_{\theta \rightarrow 0} \frac{-3 \sin 3\theta}{-\sin \theta} \stackrel{H}{=} \lim_{\theta \rightarrow 0} \frac{-9 \cos 3\theta}{-\cos \theta} = 9$, which is the same slope when $\theta = \pi$.

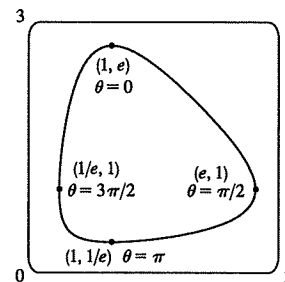
Thus, the curve has horizontal tangents at $(\frac{1}{2}, -1)$ and $(-\frac{1}{2}, 1)$, and there are no vertical tangents.

24. $x = e^{\sin \theta}$, $y = e^{\cos \theta}$. The whole curve is traced out for $0 \leq \theta < 2\pi$.

$$\frac{dy}{d\theta} = -\sin \theta e^{\cos \theta}, \text{ so } \frac{dy}{d\theta} = 0 \Leftrightarrow \sin \theta = 0 \Leftrightarrow \theta = 0 \text{ or } \pi \Leftrightarrow$$

$$(x, y) = (1, e) \text{ or } (1, 1/e). \quad \frac{dx}{d\theta} = \cos \theta e^{\sin \theta}, \text{ so } \frac{dx}{d\theta} = 0 \Leftrightarrow \cos \theta = 0 \Leftrightarrow$$

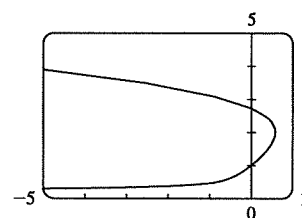
$$\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \Leftrightarrow (x, y) = (e, 1) \text{ or } (1/e, 1). \text{ The curve has horizontal tangents at } (1, e) \text{ and } (1, 1/e), \text{ and vertical tangents at } (e, 1) \text{ and } (1/e, 1).$$



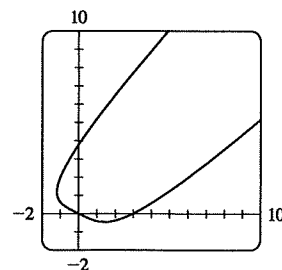
25. From the graph, it appears that the rightmost point on the curve $x = t - t^6$, $y = e^t$ is about $(0.6, 2)$. To find the exact coordinates, we find the value of t for which the graph has a vertical tangent, that is, $0 = dx/dt = 1 - 6t^5 \Leftrightarrow t = 1/\sqrt[5]{6}$.

Hence, the rightmost point is

$$\left(1/\sqrt[5]{6} - 1/(6\sqrt[5]{6}), e^{1/\sqrt[5]{6}}\right) = \left(5 \cdot 6^{-6/5}, e^{6^{-1/5}}\right) \approx (0.58, 2.01).$$



26. From the graph, it appears that the lowest point and the leftmost point on the curve $x = t^4 - 2t$, $y = t + t^4$ are $(1.5, -0.5)$ and $(-1.2, 1.2)$, respectively. To find the exact coordinates, we solve $dy/dt = 0$ (horizontal tangents) and $dx/dt = 0$ (vertical tangents).



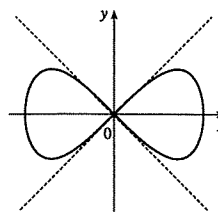
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29. $x = \cos t$, $y = \sin t \cos t$. $dx/dt = -\sin t$,

$dy/dt = -\sin^2 t + \cos^2 t = \cos 2t$. $(x, y) = (0, 0) \Leftrightarrow \cos t = 0 \Leftrightarrow t$ is an odd multiple of $\frac{\pi}{2}$. When $t = \frac{\pi}{2}$, $dx/dt = -1$ and $dy/dt = -1$, so $dy/dx = 1$.

When $t = \frac{3\pi}{2}$, $dx/dt = 1$ and $dy/dt = -1$. So $dy/dx = -1$. Thus, $y = x$ and $y = -x$ are both tangent to the curve at $(0, 0)$.



30. $x = -2 \cos t$, $y = \sin t + \sin 2t$. From the graph, it appears that the curve crosses itself at the point $(1, 0)$. If this is true, then $x = 1 \Leftrightarrow$

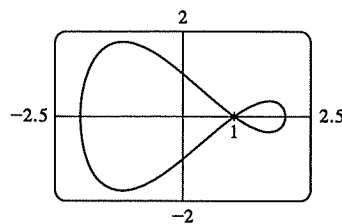
$-2 \cos t = 1 \Leftrightarrow \cos t = -\frac{1}{2} \Leftrightarrow t = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$ for $0 \leq t \leq 2\pi$.

Substituting either value of t into y gives $y = 0$, confirming that $(1, 0)$ is the

point where the curve crosses itself. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t + 2 \cos 2t}{2 \sin t}$.

When $t = \frac{2\pi}{3}$, $\frac{dy}{dx} = \frac{-1/2 + 2(-1/2)}{2(\sqrt{3}/2)} = \frac{-3/2}{\sqrt{3}} = -\frac{\sqrt{3}}{2}$, so an equation of the tangent line is $y - 0 = -\frac{\sqrt{3}}{2}(x - 1)$,

or $y = -\frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{2}$. Similarly, when $t = \frac{4\pi}{3}$, an equation of the tangent line is $y = \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{2}$.



31. $x = r\theta - d \sin \theta$, $y = r - d \cos \theta$.

(a) $\frac{dx}{d\theta} = r - d \cos \theta$, $\frac{dy}{d\theta} = d \sin \theta$, so $\frac{dy}{dx} = \frac{d \sin \theta}{r - d \cos \theta}$.

(b) If $0 < d < r$, then $|d \cos \theta| \leq d < r$, so $r - d \cos \theta \geq r - d > 0$. This shows that $dx/d\theta$ never vanishes, so the trochoid can have no vertical tangent if $d < r$.

32. $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.

(a) $\frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta$, $\frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$, so $\frac{dy}{dx} = -\frac{\sin \theta}{\cos \theta} = -\tan \theta$.

(b) The tangent is horizontal $\Leftrightarrow dy/dx = 0 \Leftrightarrow \tan \theta = 0 \Leftrightarrow \theta = n\pi \Leftrightarrow (x, y) = (\pm a, 0)$.

The tangent is vertical $\Leftrightarrow \cos \theta = 0 \Leftrightarrow \theta$ is an odd multiple of $\frac{\pi}{2} \Leftrightarrow (x, y) = (0, \pm a)$.

(c) $dy/dx = \pm 1 \Leftrightarrow \tan \theta = \pm 1 \Leftrightarrow \theta$ is an odd multiple of $\frac{\pi}{4} \Leftrightarrow (x, y) = (\pm \frac{\sqrt{2}}{4}a, \pm \frac{\sqrt{2}}{4}a)$

[All sign choices are valid.]

33. $x = 3t^2 + 1$, $y = t^3 - 1 \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{6t} = \frac{t}{2}$. The tangent line has slope $\frac{1}{2}$ when $\frac{t}{2} = \frac{1}{2} \Leftrightarrow t = 1$, so the point is $(4, 0)$.

34. $x = 3t^2 + 1$, $y = 2t^3 + 1$, $\frac{dx}{dt} = 6t$, $\frac{dy}{dt} = 6t^2$, so $\frac{dy}{dx} = \frac{6t^2}{6t} = t$ [even where $t = 0$].

So at the point corresponding to parameter value t , an equation of the tangent line is $y - (2t^3 + 1) = t[x - (3t^2 + 1)]$.

If this line is to pass through $(4, 3)$, we must have $3 - (2t^3 + 1) = t[4 - (3t^2 + 1)] \Leftrightarrow 2t^3 - 2 = 3t^3 - 3t \Leftrightarrow$

45. $x = t - 2 \sin t$, $y = 1 - 2 \cos t$, $0 \leq t \leq 4\pi$. $dx/dt = 1 - 2 \cos t$ and $dy/dt = 2 \sin t$, so

$$(dx/dt)^2 + (dy/dt)^2 = (1 - 2 \cos t)^2 + (2 \sin t)^2 = 1 - 4 \cos t + 4 \cos^2 t + 4 \sin^2 t = 5 - 4 \cos t. \text{ Thus,}$$

$$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_0^{4\pi} \sqrt{5 - 4 \cos t} dt \approx 26.7298.$$

46. $x = t \cos t$, $y = t - 5 \sin t$. $dx/dt = \cos t - t \sin t$ and $dy/dt = 1 - 5 \cos t$, so

$$(dx/dt)^2 + (dy/dt)^2 = (\cos t - t \sin t)^2 + (1 - 5 \cos t)^2. \text{ Observe that when } t = -\pi, (x, y) = (\pi, -\pi) \text{ and when } t = \pi,$$

$$(x, y) = (-\pi, \pi). \text{ Thus, } L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_{-\pi}^{\pi} \sqrt{(\cos t - t \sin t)^2 + (1 - 5 \cos t)^2} dt \approx 22.8546.$$

47. $x = \frac{2}{3}t^3$, $y = t^2 - 2$, $0 \leq t \leq 3$. $dx/dt = 2t^2$ and $dy/dt = 2t$, so $(dx/dt)^2 + (dy/dt)^2 = 4t^4 + 4t^2 = 4t^2(t^2 + 1)$.

Thus,

$$\begin{aligned} L &= \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_0^3 \sqrt{4t^2(t^2 + 1)} dt = \int_0^3 2t \sqrt{t^2 + 1} dt \\ &= \int_1^{10} \sqrt{u} du \quad [u = t^2 + 1, du = 2t dt] = \left[\frac{2}{3} u^{3/2} \right]_1^{10} = \frac{2}{3} (10^{3/2} - 1) = \frac{2}{3} (10\sqrt{10} - 1) \end{aligned}$$

48. $x = e^t - t$, $y = 4e^{t/2}$, $0 \leq t \leq 2$. $dx/dt = e^t - 1$ and $dy/dt = 2e^{t/2}$, so

$$(dx/dt)^2 + (dy/dt)^2 = (e^t - 1)^2 + (2e^{t/2})^2 = e^{2t} - 2e^t + 1 + 4e^t = e^{2t} + 2e^t + 1 = (e^t + 1)^2. \text{ Thus,}$$

$$L = \int_0^2 \sqrt{(e^t + 1)^2} dt = \int_0^2 |e^t + 1| dt = \int_0^2 (e^t + 1) dt = [e^t + t]_0^2 = (e^2 + 2) - (1 + 0) = e^2 + 1.$$

49. $x = t \sin t$, $y = t \cos t$, $0 \leq t \leq 1$. $\frac{dx}{dt} = t \cos t + \sin t$ and $\frac{dy}{dt} = -t \sin t + \cos t$, so

$$\begin{aligned} \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 &= t^2 \cos^2 t + 2t \sin t \cos t + \sin^2 t + t^2 \sin^2 t - 2t \sin t \cos t + \cos^2 t \\ &= t^2 (\cos^2 t + \sin^2 t) + \sin^2 t + \cos^2 t = t^2 + 1. \end{aligned}$$

$$\text{Thus, } L = \int_0^1 \sqrt{t^2 + 1} dt \stackrel{21}{=} \left[\frac{1}{2} t \sqrt{t^2 + 1} + \frac{1}{2} \ln(t + \sqrt{t^2 + 1}) \right]_0^1 = \frac{1}{2} \sqrt{2} + \frac{1}{2} \ln(1 + \sqrt{2}).$$

50. $x = 3 \cos t - \cos 3t$, $y = 3 \sin t - \sin 3t$, $0 \leq t \leq \pi$. $\frac{dx}{dt} = -3 \sin t + 3 \sin 3t$ and $\frac{dy}{dt} = 3 \cos t - 3 \cos 3t$, so

$$\begin{aligned} \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 &= 9 \sin^2 t - 18 \sin t \sin 3t + 9 \sin^2 3t + 9 \cos^2 t - 18 \cos t \cos 3t + 9 \cos^2 3t \\ &= 9(\cos^2 t + \sin^2 t) - 18(\cos t \cos 3t + \sin t \sin 3t) + 9(\cos^2 3t + \sin^2 3t) \\ &= 9(1) - 18 \cos(t - 3t) + 9(1) = 18 - 18 \cos(-2t) = 18(1 - \cos 2t) \\ &= 18[1 - (1 - 2 \sin^2 t)] = 36 \sin^2 t. \end{aligned}$$

$$\text{Thus, } L = \int_0^\pi \sqrt{36 \sin^2 t} dt = 6 \int_0^\pi |\sin t| dt = 6 \int_0^\pi \sin t dt = -6 [\cos t]_0^\pi = -6(-1 - 1) = 12.$$