

HW #7 SEC 10.4 SOLUTIONS

7. $r = 4 + 3 \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \frac{1}{2} (4 + 3 \sin \theta)^2 d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} (16 + 24 \sin \theta + 9 \sin^2 \theta) d\theta \\ &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} (16 + 9 \sin^2 \theta) d\theta \quad [\text{by Theorem 5.5.7}] \\ &= \frac{1}{2} \cdot 2 \int_0^{\pi/2} [16 + 9 \cdot \frac{1}{2} (1 - \cos 2\theta)] d\theta \quad [\text{by Theorem 5.5.7}] \\ &= \int_0^{\pi/2} \left(\frac{41}{2} - \frac{9}{2} \cos 2\theta \right) d\theta = \left[\frac{41}{2} \theta - \frac{9}{4} \sin 2\theta \right]_0^{\pi/2} = \left(\frac{41\pi}{4} - 0 \right) - (0 - 0) = \frac{41\pi}{4} \end{aligned}$$

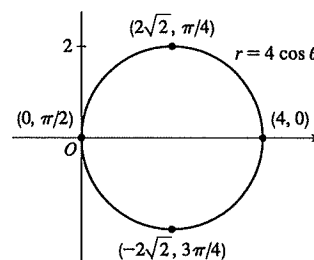
8. $r = \sqrt{\ln \theta}$, $1 \leq \theta \leq 2\pi$.

$$\begin{aligned} A &= \int_1^{2\pi} \frac{1}{2} (\sqrt{\ln \theta})^2 d\theta = \int_1^{2\pi} \frac{1}{2} \ln \theta d\theta = \left[\frac{1}{2} \theta \ln \theta \right]_1^{2\pi} - \int_1^{2\pi} \frac{1}{2} d\theta \quad \left[\begin{array}{l} u = \ln \theta, \quad dv = \frac{1}{2} d\theta \\ du = (1/\theta) d\theta, \quad v = \frac{1}{2} \theta \end{array} \right] \\ &= [\pi \ln(2\pi) - 0] - \left[\frac{1}{2} \theta \right]_1^{2\pi} = \pi \ln(2\pi) - \pi + \frac{1}{2} \end{aligned}$$

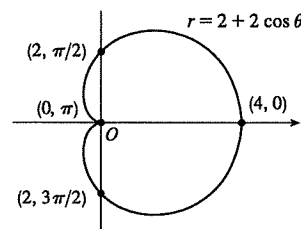
9. The area is bounded by $r = 4 \cos \theta$ for $\theta = 0$ to $\theta = \pi$.

$$\begin{aligned} A &= \int_0^\pi \frac{1}{2} r^2 d\theta = \int_0^\pi \frac{1}{2} (4 \cos \theta)^2 d\theta = \int_0^\pi 8 \cos^2 \theta d\theta \\ &= 8 \int_0^\pi \frac{1}{2} (1 + \cos 2\theta) d\theta = 4 \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^\pi = 4\pi \end{aligned}$$

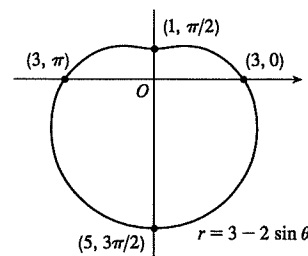
Also, note that this is a circle with radius 2, so its area is $\pi(2)^2 = 4\pi$.



$$\begin{aligned} 10. A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (2 + 2 \cos \theta)^2 d\theta \\ &= \int_0^{2\pi} \frac{1}{2} (4 + 8 \cos \theta + 4 \cos^2 \theta) d\theta \\ &= \int_0^{2\pi} \frac{1}{2} \left[4 + 8 \cos \theta + 4 \cdot \frac{1}{2} (1 + \cos 2\theta) \right] d\theta \\ &= \int_0^{2\pi} (3 + 4 \cos \theta + \cos 2\theta) d\theta = \left[3\theta + 4 \sin \theta + \frac{1}{2} \sin 2\theta \right]_0^{2\pi} = 6\pi \end{aligned}$$



$$\begin{aligned} 11. A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (3 - 2 \sin \theta)^2 d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (9 - 12 \sin \theta + 4 \sin^2 \theta) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left[9 - 12 \sin \theta + 4 \cdot \frac{1}{2} (1 - \cos 2\theta) \right] d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (11 - 12 \sin \theta - 2 \cos 2\theta) d\theta = \frac{1}{2} \left[11\theta + 12 \cos \theta - \sin 2\theta \right]_0^{2\pi} \\ &= \frac{1}{2} [(22\pi + 12) - 12] = 11\pi \end{aligned}$$

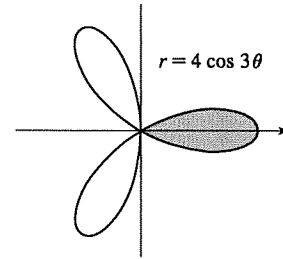


17. The curve passes through the pole when $r = 0 \Rightarrow 4 \cos 3\theta = 0 \Rightarrow \cos 3\theta = 0 \Rightarrow 3\theta = \frac{\pi}{2} + \pi n \Rightarrow$

$\theta = \frac{\pi}{6} + \frac{\pi}{3}n$. The part of the shaded loop above the polar axis is traced out for

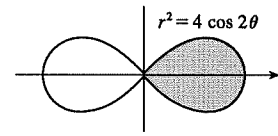
$\theta = 0$ to $\theta = \pi/6$, so we'll use $-\pi/6$ and $\pi/6$ as our limits of integration.

$$\begin{aligned} A &= \int_{-\pi/6}^{\pi/6} \frac{1}{2} (4 \cos 3\theta)^2 d\theta = 2 \int_0^{\pi/6} \frac{1}{2} (16 \cos^2 3\theta) d\theta \\ &= 16 \int_0^{\pi/6} \cos^2 3\theta d\theta = 8 \left[\theta + \frac{1}{6} \sin 6\theta \right]_0^{\pi/6} = 8 \left(\frac{\pi}{6} \right) = \frac{4}{3}\pi \end{aligned}$$



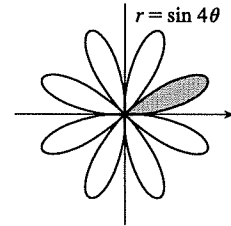
18. The curve given by $r^2 = 4 \cos 2\theta$ passes through the pole when $r = 0 \Rightarrow 4 \cos 2\theta = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow$
 $2\theta = \frac{\pi}{2} + \pi n \Rightarrow \theta = \frac{\pi}{4} + \frac{\pi}{2}n$. The part of the shaded loop above the polar axis is traced out for $\theta = 0$ to $\theta = \pi/4$,
 so we'll use $-\pi/4$ to $\pi/4$ as our limits of integration.

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2} (4 \cos 2\theta) d\theta = 2 \int_0^{\pi/4} 2 \cos 2\theta d\theta = 2 \left[\sin 2\theta \right]_0^{\pi/4} \\ &= 2 \sin \frac{\pi}{2} = 2(1) = 2 \end{aligned}$$



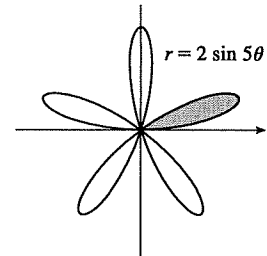
19. $r = 0 \Rightarrow \sin 4\theta = 0 \Rightarrow 4\theta = \pi n \Rightarrow \theta = \frac{\pi}{4}n$.

$$\begin{aligned} A &= \int_0^{\pi/4} \frac{1}{2} (\sin 4\theta)^2 d\theta = \frac{1}{2} \int_0^{\pi/4} \sin^2 4\theta d\theta = \frac{1}{2} \int_0^{\pi/4} \frac{1}{2} (1 - \cos 8\theta) d\theta \\ &= \frac{1}{4} \left[\theta - \frac{1}{8} \sin 8\theta \right]_0^{\pi/4} = \frac{1}{4} \left(\frac{\pi}{4} \right) = \frac{1}{16}\pi \end{aligned}$$

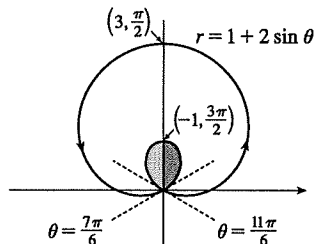
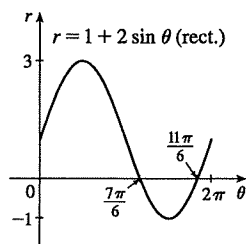


20. $r = 0 \Rightarrow 2 \sin 5\theta = 0 \Rightarrow \sin 5\theta = 0 \Rightarrow 5\theta = \pi n \Rightarrow \theta = \frac{\pi}{5}n$.

$$\begin{aligned} A &= \int_0^{\pi/5} \frac{1}{2} (2 \sin 5\theta)^2 d\theta = \frac{1}{2} \int_0^{\pi/5} 4 \sin^2 5\theta d\theta \\ &= 2 \int_0^{\pi/5} \frac{1}{2} (1 - \cos 10\theta) d\theta = \left[\theta - \frac{1}{10} \sin 10\theta \right]_0^{\pi/5} = \frac{\pi}{5} \end{aligned}$$



21.



This is a limaçon, with inner loop traced out between $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ [found by solving $r = 0$].

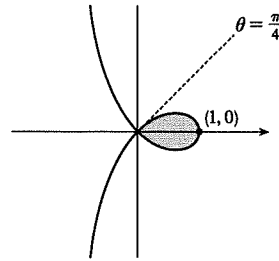
$$\begin{aligned} A &= 2 \int_{7\pi/6}^{3\pi/2} \frac{1}{2} (1 + 2 \sin \theta)^2 d\theta = \int_{7\pi/6}^{3\pi/2} (1 + 4 \sin \theta + 4 \sin^2 \theta) d\theta = \int_{7\pi/6}^{3\pi/2} \left[1 + 4 \sin \theta + 4 \cdot \frac{1}{2} (1 - \cos 2\theta) \right] d\theta \\ &= \left[\theta - 4 \cos \theta + 2\theta - \sin 2\theta \right]_{7\pi/6}^{3\pi/2} = \left(\frac{9\pi}{2} \right) - \left(\frac{7\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2} \right) = \pi - \frac{3\sqrt{3}}{2} \end{aligned}$$

22. To determine when the strophoid $r = 2 \cos \theta - \sec \theta$ passes through the pole, we solve

$$r = 0 \Rightarrow 2 \cos \theta - \frac{1}{\cos \theta} = 0 \Rightarrow 2 \cos^2 \theta - 1 = 0 \Rightarrow \cos^2 \theta = \frac{1}{2} \Rightarrow$$

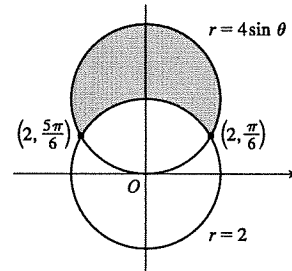
$$\cos \theta = \pm \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \text{ or } \theta = \frac{3\pi}{4} \text{ for } 0 \leq \theta \leq \pi \text{ with } \theta \neq \frac{\pi}{2}.$$

$$\begin{aligned} A &= 2 \int_0^{\pi/4} \frac{1}{2} (2 \cos \theta - \sec \theta)^2 d\theta = \int_0^{\pi/4} (4 \cos^2 \theta - 4 + \sec^2 \theta) d\theta \\ &= \int_0^{\pi/4} \left[4 \cdot \frac{1}{2} (1 + \cos 2\theta) - 4 + \sec^2 \theta \right] d\theta = \int_0^{\pi/4} (-2 + 2 \cos 2\theta + \sec^2 \theta) d\theta \\ &= [-2\theta + \sin 2\theta + \tan \theta]_0^{\pi/4} = \left(-\frac{\pi}{2} + 1 + 1\right) - 0 = 2 - \frac{\pi}{2} \end{aligned}$$



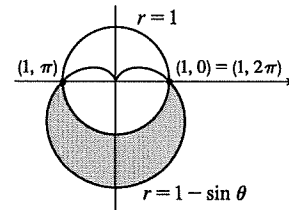
23. $4 \sin \theta = 2 \Rightarrow \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \Rightarrow$

$$\begin{aligned} A &= \int_{\pi/6}^{5\pi/6} \frac{1}{2} [(4 \sin \theta)^2 - 2^2] d\theta = 2 \int_{\pi/6}^{5\pi/6} \frac{1}{2} (16 \sin^2 \theta - 4) d\theta \\ &= \int_{\pi/6}^{5\pi/6} [16 \cdot \frac{1}{2} (1 - \cos 2\theta) - 4] d\theta = \int_{\pi/6}^{5\pi/6} (4 - 8 \cos 2\theta) d\theta \\ &= [4\theta - 4 \sin 2\theta]_{\pi/6}^{5\pi/6} = (2\pi - 0) - \left(\frac{2\pi}{3} - 2\sqrt{3}\right) = \frac{4\pi}{3} + 2\sqrt{3} \end{aligned}$$



24. $1 - \sin \theta = 1 \Rightarrow \sin \theta = 0 \Rightarrow \theta = 0 \text{ or } \pi \Rightarrow$

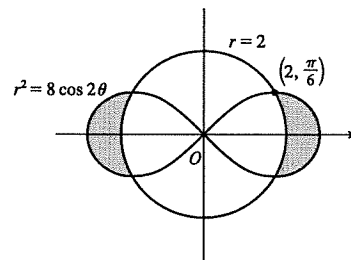
$$\begin{aligned} A &= \int_{\pi}^{2\pi} \frac{1}{2} [(1 - \sin \theta)^2 - 1] d\theta = \frac{1}{2} \int_{\pi}^{2\pi} (\sin^2 \theta - 2 \sin \theta) d\theta \\ &= \frac{1}{4} \int_{\pi}^{2\pi} (1 - \cos 2\theta - 4 \sin \theta) d\theta = \frac{1}{4} \left[\theta - \frac{1}{2} \sin 2\theta + 4 \cos \theta \right]_{\pi}^{2\pi} \\ &= \frac{1}{4} \pi + 2 \end{aligned}$$



25. To find the area inside the lemniscate $r^2 = 8 \cos 2\theta$ and outside the circle $r = 2$,

we first note that the two curves intersect when $r^2 = 8 \cos 2\theta$ and $r = 2$, that is, when $\cos 2\theta = \frac{1}{2}$. For $-\pi < \theta \leq \pi$, $\cos 2\theta = \frac{1}{2} \Leftrightarrow 2\theta = \pm\pi/3$ or $\pm 5\pi/3 \Leftrightarrow \theta = \pm\pi/6$ or $\pm 5\pi/6$. The figure shows that the desired area is 4 times the area between the curves from 0 to $\pi/6$. Thus,

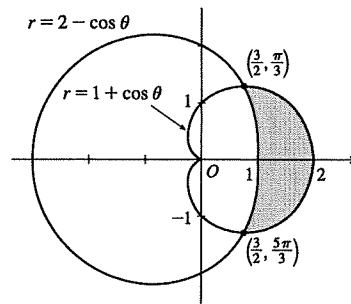
$$\begin{aligned} A &= 4 \int_0^{\pi/6} \left[\frac{1}{2} (8 \cos 2\theta) - \frac{1}{2} (2)^2 \right] d\theta = 8 \int_0^{\pi/6} (2 \cos 2\theta - 1) d\theta \\ &= 8 [\sin 2\theta - \theta]_0^{\pi/6} = 8 \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6} \right) = 4\sqrt{3} - 4\pi/3 \end{aligned}$$



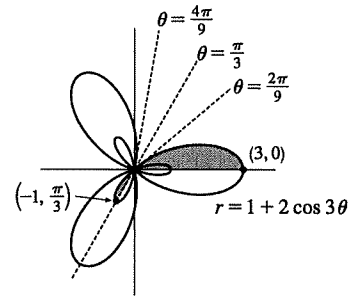
26. $1 + \cos \theta = 2 - \cos \theta \Rightarrow 2 \cos \theta = 1 \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow$

$$\theta = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}.$$

$$\begin{aligned} A &= 2 \int_0^{\pi/3} \frac{1}{2} [(1 + \cos \theta)^2 - (2 - \cos \theta)^2] d\theta \quad [\text{by symmetry}] \\ &= \int_0^{\pi/3} (1 + 2 \cos \theta + \cos^2 \theta - 4 + 4 \cos \theta - \cos^2 \theta) d\theta \\ &= \int_0^{\pi/3} (6 \cos \theta - 3) d\theta = [6 \sin \theta - 3\theta]_0^{\pi/3} = 6 \left(\frac{\sqrt{3}}{2} \right) - 3 \left(\frac{\pi}{3} \right) \\ &= 3\sqrt{3} - \pi \end{aligned}$$



36. $r = 0 \Rightarrow 1 + 2 \cos 3\theta = 0 \Rightarrow \cos 3\theta = -\frac{1}{2} \Rightarrow 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ [for $0 \leq 3\theta \leq 2\pi$] $\Rightarrow \theta = \frac{2\pi}{9}, \frac{4\pi}{9}$. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/9$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/9$ to $\theta = \pi/3$), and then double that difference to obtain the desired area.



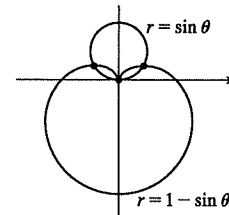
$$A = 2 \left[\int_0^{2\pi/9} \frac{1}{2} (1 + 2 \cos 3\theta)^2 d\theta - \int_{2\pi/9}^{\pi/3} \frac{1}{2} (1 + 2 \cos 3\theta)^2 d\theta \right]$$

$$\begin{aligned} \text{Now } r^2 &= (1 + 2 \cos 3\theta)^2 = 1 + 4 \cos 3\theta + 4 \cos^2 3\theta = 1 + 4 \cos 3\theta + 4 \cdot \frac{1}{2} (1 + \cos 6\theta) \\ &= 1 + 4 \cos 3\theta + 2 + 2 \cos 6\theta = 3 + 4 \cos 3\theta + 2 \cos 6\theta \end{aligned}$$

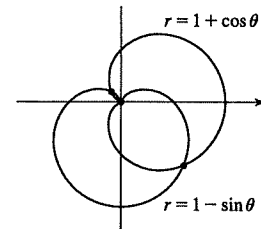
and $\int r^2 d\theta = 3\theta + \frac{4}{3} \sin 3\theta + \frac{1}{3} \sin 6\theta + C$, so

$$\begin{aligned} A &= \left[3\theta + \frac{4}{3} \sin 3\theta + \frac{1}{3} \sin 6\theta \right]_0^{2\pi/9} - \left[3\theta + \frac{4}{3} \sin 3\theta + \frac{1}{3} \sin 6\theta \right]_{2\pi/9}^{\pi/3} \\ &= \left[\left(\frac{2\pi}{3} + \frac{4}{3} \cdot \frac{\sqrt{3}}{2} + \frac{1}{3} \cdot \frac{-\sqrt{3}}{2} \right) - 0 \right] - \left[(\pi + 0 + 0) - \left(\frac{2\pi}{3} + \frac{4}{3} \cdot \frac{\sqrt{3}}{2} + \frac{1}{3} \cdot \frac{-\sqrt{3}}{2} \right) \right] \\ &= \frac{4\pi}{3} + \frac{4}{3} \sqrt{3} - \frac{1}{3} \sqrt{3} - \pi = \frac{\pi}{3} + \sqrt{3} \end{aligned}$$

37. The pole is a point of intersection. $\sin \theta = 1 - \sin \theta \Rightarrow 2 \sin \theta = 1 \Rightarrow \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. So the other points of intersection are $(\frac{1}{2}, \frac{\pi}{6})$ and $(\frac{1}{2}, \frac{5\pi}{6})$.



38. The pole is a point of intersection. $1 + \cos \theta = 1 - \sin \theta \Rightarrow \cos \theta = -\sin \theta \Rightarrow \frac{\cos \theta}{\sin \theta} = -1 \Rightarrow \cot \theta = -1 \Rightarrow \theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. So the other points of intersection are $(1 - \frac{1}{2}\sqrt{2}, \frac{3\pi}{4})$ and $(1 + \frac{1}{2}\sqrt{2}, \frac{7\pi}{4})$.



39. $2 \sin 2\theta = 1 \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \text{ or } \frac{17\pi}{6}$.

By symmetry, the eight points of intersection are given by

$(1, \theta)$, where $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \text{ and } \frac{17\pi}{12}$, and

$(-1, \theta)$, where $\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \text{ and } \frac{23\pi}{12}$.

[There are many ways to describe these points.]

