HW #5, SECTION 11.4 SOLUTIONS

(b) Use the Limit Comparison Test with $a_n = \dfrac{n^2 - n}{n^3 + 2}$ and $b_n = \dfrac{1}{n}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2 - n}{n^3 + 2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^3 - n^2}{n^3 + 2} = \lim_{n\to\infty} \frac{1 - 1/n}{1 + 2/n^3} = \frac{1 - 0}{1 + 0} = 1 > 0$$

Since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n}$ is a divergent (partial) p-series $[p = 1 \leq 1]$, the series $\sum\limits_{n=2}^{\infty} \dfrac{n^2 - n}{n^3 + 2}$ also diverges.

5. An inequality can be used to show that a series converges if its general term can be shown to be less than or equal to the general term of a known convergent series. The only inequality that satisfies this condition is given in part (c) since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ is a convergent p-series $[p = 2 > 1]$.

6. An inequality can be used to show that a series diverges if its general term can be shown to be greater than or equal to the general term of a known divergent series. The only inequality that satisfies this condition is given in part (c) since $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n} = \dfrac{1}{2} \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ is half of the harmonic series, which is divergent.

7. $\dfrac{1}{n^3 + 8} < \dfrac{1}{n^3}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3 + 8}$ converges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$, which converges because it is a p-series with $p = 3 > 1$.

8. $\dfrac{1}{\sqrt{n} - 1} > \dfrac{1}{\sqrt{n}}$ for all $n \geq 2$, so $\sum\limits_{n=2}^{\infty} \dfrac{1}{\sqrt{n} - 1}$ diverges by direct comparison with $\sum\limits_{n=2}^{\infty} \dfrac{1}{\sqrt{n}}$, which diverges because it is a p-series with $p = \frac{1}{2} \leq 1$.

9. $\dfrac{n + 1}{n\sqrt{n}} > \dfrac{n}{n\sqrt{n}} = \dfrac{1}{\sqrt{n}}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n + 1}{n\sqrt{n}}$ diverges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$, which diverges because it is a p-series with $p = \frac{1}{2} \leq 1$.

10. $\dfrac{n - 1}{n^3 + 1} < \dfrac{n}{n^3 + 1} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n - 1}{n^3 + 1}$ converges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

11. $\dfrac{9^n}{3 + 10^n} < \dfrac{9^n}{10^n} = \left(\dfrac{9}{10}\right)^n$ for all $n \geq 1$. $\sum\limits_{n=1}^{\infty} \left(\frac{9}{10}\right)^n$ is a convergent geometric series $\left(|r| = \frac{9}{10} < 1\right)$, so $\sum\limits_{n=1}^{\infty} \dfrac{9^n}{3 + 10^n}$ converges by the Direct Comparison Test.

12. $\dfrac{6^n}{5^n - 1} > \dfrac{6^n}{5^n} = \left(\dfrac{6}{5}\right)^n$ for all $n \geq 1$. $\sum\limits_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$ is a divergent geometric series $\left(|r| = \frac{6}{5} > 1\right)$, so $\sum\limits_{n=1}^{\infty} \dfrac{6^n}{5^n - 1}$ diverges by the Direct Comparison Test.

13. For $n \geq 2$, $\ln n < n$, so $\dfrac{1}{\ln n} > \dfrac{1}{n}$. Thus, $\sum\limits_{n=2}^{\infty} \dfrac{1}{\ln n}$ diverges by direct comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$, which diverges because it is a p-series with $p = 1 \leq 1$ (the harmonic series).

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23. Use the Limit Comparison Test with $a_n = \dfrac{n+1}{n^3+n}$ and $b_n = \dfrac{1}{n^2}$:

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{(n+1)n^2}{n(n^2+1)} = \lim\limits_{n\to\infty} \dfrac{n^2+n}{n^2+1} = \lim\limits_{n\to\infty} \dfrac{1+1/n}{1+1/n^2} = 1 > 0$. Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ is a convergent p-series $[p = 2 > 1]$, the series $\sum\limits_{n=1}^{\infty} \dfrac{n+1}{n^3+n}$ also converges.

24. Use the Limit Comparison Test with $a_n = \dfrac{n^2+n+1}{n^4+n^2}$ and $b_n = \dfrac{1}{n^2}$:

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{(n^2+n+1)n^2}{n^2(n^2+1)} = \lim\limits_{n\to\infty} \dfrac{n^2+n+1}{n^2+1} = \lim\limits_{n\to\infty} \dfrac{1+1/n+1/n^2}{1+1/n^2} = 1 > 0$. Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ is a convergent p-series $[p = 2 > 1]$, the series $\sum\limits_{n=1}^{\infty} \dfrac{n^2+n+1}{n^4+n^2}$ also converges.

25. Use the Limit Comparison Test with $a_n = \dfrac{\sqrt{1+n}}{2+n}$ and $b_n = \dfrac{1}{\sqrt{n}}$:

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\sqrt{1+n}\sqrt{n}}{2+n} = \lim\limits_{n\to\infty} \dfrac{\sqrt{n^2+n}/\sqrt{n^2}}{(2+n)/n} = \lim\limits_{n\to\infty} \dfrac{\sqrt{1+1/n}}{2/n+1} = 1 > 0$. Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ is a divergent p-series $[p = \frac{1}{2} \le 1]$, the series $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{1+n}}{2+n}$ also diverges.

26. Use the Limit Comparison Test with $a_n = \dfrac{n+2}{(n+1)^3}$ and $b_n = \dfrac{1}{n^2}$:

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^2(n+2)}{(n+1)^3} = \lim\limits_{n\to\infty} \dfrac{1+\frac{2}{n}}{\left(1+\frac{1}{n}\right)^3} = 1 > 0$. Since $\sum\limits_{n=3}^{\infty} \dfrac{1}{n^2}$ is a convergent (partial) p-series $[p = 2 > 1]$, the series $\sum\limits_{n=3}^{\infty} \dfrac{n+2}{(n+1)^3}$ also converges.

27. Use the Limit Comparison Test with $a_n = \dfrac{5+2n}{(1+n^2)^2}$ and $b_n = \dfrac{1}{n^3}$:

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^3(5+2n)}{(1+n^2)^2} = \lim\limits_{n\to\infty} \dfrac{5n^3+2n^4}{(1+n^2)^2} \cdot \dfrac{1/n^4}{1/(n^2)^2} = \lim\limits_{n\to\infty} \dfrac{\frac{5}{n}+2}{\left(\frac{1}{n^2}+1\right)^2} = 2 > 0$. Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$ is a convergent p-series $[p = 3 > 1]$, the series $\sum\limits_{n=1}^{\infty} \dfrac{5+2n}{(1+n^2)^2}$ also converges.

28. $\dfrac{n+3^n}{n+2^n} > \dfrac{3^n}{n+2^n} > \dfrac{3^n}{2^n+2^n} = \dfrac{3^n}{2\cdot 2^n} = \dfrac{1}{2}\left(\dfrac{3}{2}\right)^n$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{n+3^n}{n+2^n}$ diverges by direct comparison with $\dfrac{1}{2}\sum\limits_{n=1}^{\infty}\left(\dfrac{3}{2}\right)^n$, which is a constant multiple of a divergent geometric series $[|r| = \frac{3}{2} > 1]$. *Or:* Use the Limit Comparison Test with $a_n = \dfrac{n+3^n}{n+2^n}$ and $b_n = \left(\dfrac{3}{2}\right)^n$.

29. $\dfrac{e^n+1}{ne^n+1} \ge \dfrac{e^n+1}{ne^n+n} = \dfrac{e^n+1}{n(e^n+1)} = \dfrac{1}{n}$ for $n \ge 1$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{e^n+1}{ne^n+1}$ diverges by direct comparison with the divergent harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$. *Or:* Use the Limit Comparison Test with $a_n = \dfrac{e^n+1}{ne^n+1}$ and $b_n = \dfrac{1}{n}$.

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