HW #5, SECTION 11.6 SOLUTIONS

11.6 The Ratio and Root Tests

1. (a) Since $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 8 > 1$, part (b) of the Ratio Test tells us that the series $\sum a_n$ is divergent.

(b) Since $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 0.8 < 1$, part (a) of the Ratio Test tells us that the series $\sum a_n$ is absolutely convergent (and therefore convergent).

(c) Since $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test fails and the series $\sum a_n$ might converge or it might diverge.

2. $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\dfrac{1}{a_n/a_{n+1}}\right| = \left|\dfrac{1}{2}\right| = \dfrac{1}{2} < 1$ Thus, the series $\sum a_n$ is absolutely convergent (and therefore convergent) by the Ratio Test.

3. $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\dfrac{n+1}{5^{n+1}}\cdot\dfrac{5^n}{n}\right| = \lim_{n\to\infty}\left|\dfrac{1}{5}\cdot\dfrac{n+1}{n}\right| = \dfrac{1}{5}\lim_{n\to\infty}\dfrac{1+1/n}{1} = \dfrac{1}{5}(1) = \dfrac{1}{5} < 1$, so the series $\sum_{n=1}^{\infty}\dfrac{n}{5^n}$ is absolutely convergent by the Ratio Test.

4. $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\dfrac{(-2)^{n+1}}{(n+1)^2}\cdot\dfrac{n^2}{(-2)^n}\right| = \lim_{n\to\infty}\left|(-2)\dfrac{n^2}{(n+1)^2}\right| = 2\lim_{n\to\infty}\dfrac{1}{(1+1/n)^2} = 2(1) = 2 > 1$, so the series $\sum_{n=1}^{\infty}\dfrac{(-2)^n}{n^2}$ is divergent by the Ratio Test.

5. $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\dfrac{(-1)^n 3^{n+1}}{2^{n+1}(n+1)^3}\cdot\dfrac{2^n n^3}{(-1)^{n-1}3^n}\right| = \lim_{n\to\infty}\left|\left(-\dfrac{3}{2}\right)\dfrac{n^3}{(n+1)^3}\right| = \dfrac{3}{2}\lim_{n\to\infty}\dfrac{1}{(1+1/n)^3} = \dfrac{3}{2}(1) = \dfrac{3}{2} > 1$,

so the series $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{3^n}{2^n n^3}$ is divergent by the Ratio Test.

6. $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\dfrac{(-3)^{n+1}}{[2(n+1)+1]!}\cdot\dfrac{(2n+1)!}{(-3)^n}\right| = \lim_{n\to\infty}\left|(-3)\dfrac{1}{(2n+3)(2n+2)}\right| = 3\lim_{n\to\infty}\dfrac{1}{(2n+3)(2n+2)}$

$= 3(0) = 0 < 1$

so the series $\sum_{n=0}^{\infty}\dfrac{(-3)^n}{(2n+1)!}$ is absolutely convergent by the Ratio Test.

7. $\lim_{k\to\infty}\left|\dfrac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\dfrac{1}{(k+1)!}\cdot\dfrac{k!}{1}\right| = \lim_{k\to\infty}\dfrac{1}{k+1} = 0 < 1$, so the series $\sum_{k=1}^{\infty}\dfrac{1}{k!}$ is absolutely convergent by the Ratio Test.

Since the terms of this series are positive, absolute convergence is the same as convergence.

8. $\lim_{k\to\infty}\left|\dfrac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\dfrac{(k+1)e^{-(k+1)}}{ke^{-k}}\right| = \lim_{k\to\infty}\left(\dfrac{k+1}{k}\cdot e^{-1}\right) = \dfrac{1}{e}\lim_{k\to\infty}\dfrac{1+1/k}{1} = \dfrac{1}{e}(1) = \dfrac{1}{e} < 1$, so the series $\sum_{k=1}^{\infty}ke^{-k}$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

9. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left[\dfrac{10^{n+1}}{(n+2)\,4^{2n+3}}\cdot\dfrac{(n+1)\,4^{2n+1}}{10^n}\right] = \lim\limits_{n\to\infty}\left(\dfrac{10}{4^2}\cdot\dfrac{n+1}{n+2}\right) = \dfrac{5}{8} < 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{10^n}{(n+1)4^{2n+1}}$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

10. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left[\dfrac{(n+1)!}{100^{n+1}}\cdot\dfrac{100^n}{n!}\right] = \lim\limits_{n\to\infty}\dfrac{n+1}{100} = \infty$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{n!}{100^n}$ diverges by the Ratio Test.

11. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)\pi^{n+1}}{(-3)^n}\cdot\dfrac{(-3)^{n-1}}{n\pi^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{\pi}{-3}\cdot\dfrac{n+1}{n}\right| = \dfrac{\pi}{3}\lim\limits_{n\to\infty}\dfrac{1+1/n}{1} = \dfrac{\pi}{3}(1) = \dfrac{\pi}{3} > 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{n\pi^n}{(-3)^{n-1}}$ diverges by the Ratio Test. *Or:* Since $\lim\limits_{n\to\infty}|a_n| = \infty$, the series diverges by the Test for Divergence.

12. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)^{10}}{(-10)^{n+2}}\cdot\dfrac{(-10)^{n+1}}{n^{10}}\right| = \lim\limits_{n\to\infty}\left|\dfrac{1}{-10}\left(\dfrac{n+1}{n}\right)^{10}\right| = \dfrac{1}{10}\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{10} = \dfrac{1}{10}(1) = \dfrac{1}{10} < 1$,

so the series $\sum\limits_{n=1}^{\infty}\dfrac{n^{10}}{(-10)^{n+1}}$ is absolutely convergent by the Ratio Test.

13. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{\cos[(n+1)\pi/3]}{(n+1)!}\cdot\dfrac{n!}{\cos(n\pi/3)}\right| = \lim\limits_{n\to\infty}\left|\dfrac{\cos[(n+1)\pi/3]}{(n+1)\cos(n\pi/3)}\right| = \lim\limits_{n\to\infty}\dfrac{c}{n+1} = 0 < 1$ (where $0 < c \le 2$ for all positive integers n), so the series $\sum\limits_{n=1}^{\infty}\dfrac{\cos(n\pi/3)}{n!}$ is absolutely convergent by the Ratio Test.

14. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)!}{(n+1)^{n+1}}\cdot\dfrac{n^n}{n!}\right| = \lim\limits_{n\to\infty}\dfrac{(n+1)n^n}{(n+1)^{n+1}} = \lim\limits_{n\to\infty}\dfrac{n^n}{(n+1)^n} = \lim\limits_{n\to\infty}\dfrac{1}{(1+1/n)^n} = \dfrac{1}{e} < 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{n!}{n^n}$ is absolutely convergent by the Ratio Test.

15. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)^{100}100^{n+1}}{(n+1)!}\cdot\dfrac{n!}{n^{100}100^n}\right| = \lim\limits_{n\to\infty}\dfrac{100}{n+1}\left(\dfrac{n+1}{n}\right)^{100} = \lim\limits_{n\to\infty}\dfrac{100}{n+1}\left(1+\dfrac{1}{n}\right)^{100}$

$= 0\cdot 1 = 0 < 1$

so the series $\sum\limits_{n=1}^{\infty}\dfrac{n^{100}100^n}{n!}$ is absolutely convergent by the Ratio Test.

16. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{[2(n+1)]!}{[(n+1)!]^2}\cdot\dfrac{(n!)^2}{(2n)!}\right| = \lim\limits_{n\to\infty}\dfrac{(2n+2)(2n+1)}{(n+1)(n+1)} = \lim\limits_{n\to\infty}\dfrac{(2+2/n)(2+1/n)}{(1+1/n)(1+1/n)} = \dfrac{2\cdot 2}{1\cdot 1} = 4 > 1$,

so the series $\sum\limits_{n=1}^{\infty}\dfrac{(2n)!}{(n!)^2}$ diverges by the Ratio Test.

17. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-1)^n(n+1)!}{1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)(2n+1)}\cdot\dfrac{1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)}{(-1)^{n-1}n!}\right| = \lim\limits_{n\to\infty}\dfrac{n+1}{2n+1}$

$= \lim\limits_{n\to\infty}\dfrac{1+1/n}{2+1/n} = \dfrac{1}{2} < 1$,

so the series $1 - \dfrac{2!}{1\cdot 3} + \dfrac{3!}{1\cdot 3\cdot 5} - \dfrac{4!}{1\cdot 3\cdot 5\cdot 7} + \dots + (-1)^{n-1}\dfrac{n!}{1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)} + \cdots$ is absolutely convergent by the Ratio Test.

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18. $\dfrac{2}{3}+\dfrac{2\cdot 5}{3\cdot 5}+\dfrac{2\cdot 5\cdot 8}{3\cdot 5\cdot 7}+\dfrac{2\cdot 5\cdot 8\cdot 11}{3\cdot 5\cdot 7\cdot 9}+\cdots=\sum\limits_{n=1}^{\infty}\dfrac{2\cdot 5\cdot 8\cdot 11\cdot\cdots\cdot(3n-1)}{3\cdot 5\cdot 7\cdot 9\cdot\cdots\cdot(2n+1)}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|&=\lim_{n\to\infty}\left|\frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)(3n+2)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)(2n+3)}\cdot\frac{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}\right|\\&=\lim_{n\to\infty}\frac{3n+2}{2n+3}=\lim_{n\to\infty}\frac{3+2/n}{2+3/n}=\frac{3}{2}>1,\end{aligned}$$

so the given series diverges by the Ratio Test.

19. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left|\dfrac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)(2n+2)}{(n+1)!}\cdot\dfrac{n!}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}\right|=\lim\limits_{n\to\infty}\dfrac{2n+2}{n+1}=\lim\limits_{n\to\infty}\dfrac{2(n+1)}{n+1}=2>1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{n!}$ diverges by the Ratio Test.

20. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left|\dfrac{2^{n+1}(n+1)!}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)(3n+5)}\cdot\dfrac{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}{2^n n!}\right|=\lim\limits_{n\to\infty}\dfrac{2(n+1)}{3n+5}=\dfrac{2}{3}<1$, so the series $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{2^n n!}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}$ is absolutely convergent by the Ratio Test.

21. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\dfrac{n^2+1}{2n^2+1}=\lim\limits_{n\to\infty}\dfrac{1+1/n^2}{2+1/n^2}=\dfrac{1}{2}<1$, so the series $\sum\limits_{n=1}^{\infty}\left(\dfrac{n^2+1}{2n^2+1}\right)^n$ is absolutely convergent by the Root Test.

22. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\sqrt[n]{\left|\dfrac{(-2)^n}{n^n}\right|}=\lim\limits_{n\to\infty}\dfrac{2}{n}=0<1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{(-2)^n}{n^n}$ is absolutely convergent by the Root Test.

23. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\sqrt[n]{\left|\dfrac{(-1)^{n-1}}{(\ln n)^n}\right|}=\lim\limits_{n\to\infty}\dfrac{1}{\ln n}=0<1$, so the series $\sum\limits_{n=2}^{\infty}\dfrac{(-1)^{n-1}}{(\ln n)^n}$ is absolutely convergent by the Root Test.

24.
$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{|a_n|}&=\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{-2n}{n+1}\right)^{5n}\right|}=\lim_{n\to\infty}\frac{2^5n^5}{(n+1)^5}=32\lim_{n\to\infty}\frac{1}{\left(\dfrac{n+1}{n}\right)^5}=32\lim_{n\to\infty}\frac{1}{(1+1/n)^5}\\&=32(1)=32>1,\end{aligned}$$

so the series $\sum\limits_{n=1}^{\infty}\left(\dfrac{-2n}{n+1}\right)^{5n}$ diverges by the Root Test.

25. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\sqrt[n]{\left(1+\dfrac{1}{n}\right)^{n^2}}=\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n=e>1$ [by Equation 3.6.6], so the series $\sum\limits_{n=1}^{\infty}\left(1+\dfrac{1}{n}\right)^{n^2}$ diverges by the Root Test.

26. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\sqrt[n]{|(\arctan n)^n|}=\lim\limits_{n\to\infty}\arctan n=\frac{\pi}{2}>1$, so the series $\sum\limits_{n=0}^{\infty}(\arctan n)^n$ diverges by the Root Test.

27. $\sum\limits_{n=2}^{\infty}a_n=\sum\limits_{n=2}^{\infty}\dfrac{(-1)^n\ln n}{n}=\sum\limits_{n=2}^{\infty}(-1)^n b_n$. Now $b_n=\dfrac{\ln n}{n}>0$ for $n\geq 2$, and $\{b_n\}$ is decreasing for $n\geq 3$ since $\left(\dfrac{\ln x}{x}\right)'=\dfrac{x\cdot\frac{1}{x}-\ln x\cdot 1}{x^2}=\dfrac{1-\ln x}{x^2}<0$ when $\ln x>1$ or $x>e\approx 2.7$. Also, $\lim\limits_{n\to\infty}b_n=\lim\limits_{n\to\infty}\dfrac{\ln n}{n}\overset{H}{=}\lim\limits_{n\to\infty}\dfrac{1/n}{1}=0$, SEC 11.6

36. By the recursive definition, $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\dfrac{2+\cos n}{\sqrt{n}}\right| = 0 < 1$, so the series converges absolutely by the Ratio Test.

37. The series $\sum\limits_{n=1}^{\infty}\dfrac{b_n^n\cos n\pi}{n} = \sum\limits_{n=1}^{\infty}(-1)^n\dfrac{b_n^n}{n}$, where $b_n > 0$ for $n \geq 1$ and $\lim\limits_{n\to\infty} b_n = \dfrac{1}{2}$.

$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-1)^{n+1}b_n^{n+1}}{n+1}\cdot\dfrac{n}{(-1)^n b_n^n}\right| = \lim\limits_{n\to\infty} b_n\dfrac{n}{n+1} = \dfrac{1}{2}(1) = \dfrac{1}{2} < 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{b_n^n\cos n\pi}{n}$ is absolutely convergent by the Ratio Test.

38. $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)!}{(n+1)^{n+1}b_1b_2\cdots b_nb_{n+1}}\cdot\frac{n^nb_1b_2\cdots b_n}{(-1)^n\,n!}\right| = \lim_{n\to\infty}\left|\frac{(-1)(n+1)n^n}{b_{n+1}(n+1)^{n+1}}\right| = \lim_{n\to\infty}\frac{n^n}{b_{n+1}(n+1)^n}$$
$$= \lim_{n\to\infty}\frac{1}{b_{n+1}}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\frac{1}{b_{n+1}}\left(\frac{1}{1+1/n}\right)^n = \lim_{n\to\infty}\frac{1}{b_{n+1}(1+1/n)^n} = \frac{1}{\frac{1}{2}e} = \frac{2}{e} < 1$$

so the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n\,n!}{n^nb_1b_2b_3\cdots b_n}$ is absolutely convergent by the Ratio Test.

39. (a) $\lim\limits_{n\to\infty}\left|\dfrac{1/(n+1)^3}{1/n^3}\right| = \lim\limits_{n\to\infty}\dfrac{n^3}{(n+1)^3} = \lim\limits_{n\to\infty}\dfrac{1}{(1+1/n)^3} = 1$. Inconclusive for $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^3}$.

(b) $\lim\limits_{n\to\infty}\left|\dfrac{(n+1)}{2^{n+1}}\cdot\dfrac{2^n}{n}\right| = \lim\limits_{n\to\infty}\dfrac{n+1}{2n} = \lim\limits_{n\to\infty}\left(\dfrac{1}{2}+\dfrac{1}{2n}\right) = \dfrac{1}{2}$. Conclusive (convergent) for $\sum\limits_{n=1}^{\infty}\dfrac{n}{2^n}$.

(c) $\lim\limits_{n\to\infty}\left|\dfrac{(-3)^n}{\sqrt{n+1}}\cdot\dfrac{\sqrt{n}}{(-3)^{n-1}}\right| = 3\lim\limits_{n\to\infty}\sqrt{\dfrac{n}{n+1}} = 3\lim\limits_{n\to\infty}\sqrt{\dfrac{1}{1+1/n}} = 3$. Conclusive (divergent) for $\sum\limits_{n=1}^{\infty}\dfrac{(-3)^{n-1}}{\sqrt{n}}$.

(d) $\lim\limits_{n\to\infty}\left|\dfrac{\sqrt{n+1}}{1+(n+1)^2}\cdot\dfrac{1+n^2}{\sqrt{n}}\right| = \lim\limits_{n\to\infty}\left[\sqrt{1+\dfrac{1}{n}}\cdot\dfrac{1/n^2+1}{1/n^2+(1+1/n)^2}\right] = 1$. Inconclusive for $\sum\limits_{n=1}^{\infty}\dfrac{\sqrt{n}}{1+n^2}$.

40. We use the Ratio Test for the series $\sum\limits_{n=1}^{\infty}\dfrac{(n!)^2}{(kn)!}$:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)!]^2/[k(n+1)]!}{(n!)^2/(kn)!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{[k(n+1)]\,[k(n+1)-1]\cdots[kn+1]}\right|$$

Now if $k=1$, then this is equal to $\lim\limits_{n\to\infty}\left|\dfrac{(n+1)^2}{(n+1)}\right| = \infty$, so the series diverges; if $k=2$, the limit is

$\lim\limits_{n\to\infty}\left|\dfrac{(n+1)^2}{(2n+2)(2n+1)}\right| = \dfrac{1}{4} < 1$, so the series converges, and if $k > 2$, then the highest power of n in the denominator is larger than 2, and so the limit is 0, indicating convergence. So the series converges for $k \geq 2$.

41. (a) $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)!}\cdot\dfrac{n!}{x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x}{n+1}\right| = |x|\lim\limits_{n\to\infty}\dfrac{1}{n+1} = |x|\cdot 0 = 0 < 1$, so by the Ratio Test the series $\sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!}$ converges for all x.

(b) Since the series of part (a) always converges, we must have $\lim\limits_{n\to\infty}\dfrac{x^n}{n!} = 0$ by Theorem 11.2.6.