HW #6, SEC 11.9 SOLUTIONS

3. Our goal is to write the function in the form $\frac{1}{1-r}$, and then use Equation 1 to represent the function as a sum of a power series. $f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$ with $|-x| < 1 \Leftrightarrow |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

4. $f(x) = \frac{x}{1+x} = x\left(\frac{1}{1-(-x)}\right) = x\sum_{n=0}^{\infty}(-x)^n$, or, equivalently, $\sum_{n=0}^{\infty}(-1)^n x^{n+1}$. The series converges when $|-x| < 1 \Leftrightarrow |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

5. $f(x) = \frac{1}{1-x^2} = \sum_{n=0}^{\infty}(x^2)^n = \sum_{n=0}^{\infty}x^{2n}$. The series converges when $|x^2| < 1 \Leftrightarrow |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

6. $f(x) = \frac{5}{1-4x^2} = 5\left(\frac{1}{1-4x^2}\right) = 5\sum_{n=0}^{\infty}(4x^2)^n = 5\sum_{n=0}^{\infty}4^n x^{2n}$. The series converges when $|4x^2| < 1 \Leftrightarrow |x|^2 < \frac{1}{4} \Leftrightarrow |x| < \frac{1}{2}$, so $R = \frac{1}{2}$ and $I = \left(-\frac{1}{2}, \frac{1}{2}\right)$.

7. $f(x) = \frac{2}{3-x} = \frac{2}{3}\left(\frac{1}{1-x/3}\right) = \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$ or, equivalently, $2\sum_{n=0}^{\infty}\frac{1}{3^{n+1}}x^n$. The series converges when $\left|\frac{x}{3}\right| < 1$, that is, when $|x| < 3$, so $R = 3$ and $I = (-3, 3)$.

8. $f(x) = \frac{4}{2x+3} = \frac{4}{3}\left(\frac{1}{1+2x/3}\right) = \frac{4}{3}\left(\frac{1}{1-(-2x/3)}\right) = \frac{4}{3}\sum_{n=0}^{\infty}\left(-\frac{2x}{3}\right)^n$ or, equivalently, $\sum_{n=0}^{\infty}(-1)^n\frac{2^{n+2}}{3^{n+1}}x^n$.

The series converges when $\left|-\frac{2x}{3}\right| < 1$, that is, when $|x| < \frac{3}{2}$, so $R = \frac{3}{2}$ and $I = \left(-\frac{3}{2}, \frac{3}{2}\right)$.

9. $f(x) = \frac{x^2}{x^4+16} = \frac{x^2}{16}\left(\frac{1}{1+x^4/16}\right) = \frac{x^2}{16}\left(\frac{1}{1-[-(x/2)]^4}\right) = \frac{x^2}{16}\sum_{n=0}^{\infty}\left[-\left(\frac{x}{2}\right)^4\right]^n$ or, equivalently, $\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{2^{4n+4}}$.

The series converges when $\left|-\left(\frac{x}{2}\right)^4\right| < 1 \Rightarrow \left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$, so $R = 2$ and $I = (-2, 2)$.

10. $f(x) = \frac{x}{2x^2+1} = x\left(\frac{1}{1-(-2x^2)}\right) = x\sum_{n=0}^{\infty}(-2x^2)^n$ or, equivalently, $\sum_{n=0}^{\infty}(-1)^n 2^n x^{2n+1}$. The series converges when $|-2x^2| < 1 \Rightarrow |x^2| < \frac{1}{2} \Rightarrow |x| < \frac{1}{\sqrt{2}}$, so $R = \frac{1}{\sqrt{2}}$ and $I = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

11. $$f(x) = \frac{x-1}{x+2} = \frac{x+2-3}{x+2} = 1 - \frac{3}{x+2} = 1 - \frac{3/2}{x/2+1} = 1 - \frac{3}{2}\cdot\frac{1}{1-(-x/2)}$$
$$= 1 - \frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = 1 - \frac{3}{2} - \frac{3}{2}\sum_{n=1}^{\infty}\left(-\frac{x}{2}\right)^n = -\frac{1}{2} - \sum_{n=1}^{\infty}\frac{(-1)^n 3x^n}{2^{n+1}}.$$

The geometric series $\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$ converges when $\left|-\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$, so $R = 2$ and $I = (-2, 2)$.

Alternatively, you could write $f(x) = 1 - 3\left(\frac{1}{x+2}\right)$ and use the series for $\frac{1}{x+2}$ found in Example 2.

12. $f(x) = \dfrac{x+a}{x^2+a^2} \quad [a>0] \quad = \dfrac{x}{a^2}\left[\dfrac{1}{1-(-x^2/a^2)}\right] + \dfrac{a}{a^2}\left[\dfrac{1}{1-(-x^2/a^2)}\right]$

$$= \frac{x}{a^2}\sum_{n=0}^{\infty}\left(-\frac{x^2}{a^2}\right)^n + \frac{1}{a}\sum_{n=0}^{\infty}\left(-\frac{x^2}{a^2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{a^{2n+2}} + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{a^{2n+1}}.$$

The geometric series $\sum\limits_{n=0}^{\infty}\left(-\dfrac{x^2}{a^2}\right)^n$ converges when $\left|-\dfrac{x^2}{a^2}\right| < 1 \;\Leftrightarrow\; |x| < a$, so $R = a$ and $I = (-a, a)$.

13. $f(x) = \dfrac{2x-4}{x^2-4x+3} = \dfrac{2x-4}{(x-1)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-3} \;\Rightarrow\; 2x-4 = A(x-3) + B(x-1)$. Let $x = 1$ to get $-2 = -2A \;\Leftrightarrow\; A = 1$ and $x = 3$ to get $2 = 2B \;\Leftrightarrow\; B = 1$. Thus,

$$\frac{2x-4}{x^2-4x+3} = \frac{1}{x-1} + \frac{1}{x-3} = \frac{-1}{1-x} + \frac{1}{-3}\left[\frac{1}{1-(x/3)}\right] = -\sum_{n=0}^{\infty}x^n - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\left(-1-\frac{1}{3^{n+1}}\right)x^n.$$

We represented f as the sum of two geometric series; the first converges for $x \in (-1, 1)$ and the second converges for $x \in (-3, 3)$. Thus, the sum converges for $x \in (-1, 1) = I$.

14. $f(x) = \dfrac{2x+3}{x^2+3x+2} = \dfrac{2x+3}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} \;\Rightarrow\; 2x+3 = A(x+2) + B(x+1)$. Let $x = -1$ to get $1 = A$ and $x = -2$ to get $-1 = -B \;\Leftrightarrow\; B = 1$. Thus,

$$\frac{2x+3}{x^2+3x+2} = \frac{1}{x+1} + \frac{1}{x+2} = \frac{1}{1-(-x)} + \frac{1}{2}\left[\frac{1}{1-(-x/2)}\right]$$
$$= \sum_{n=0}^{\infty}(-x)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\left[(-1)^n\left(1+\frac{1}{2^{n+1}}\right)\right]x^n$$

We represented f as the sum of two geometric series; the first converges for $x \in (-1, 1)$ and the second converges for $x \in (-2, 2)$. Thus, the sum converges for $x \in (-1, 1) = I$.

15. (a) $f(x) = \dfrac{1}{(1+x)^2} = \dfrac{d}{dx}\left(\dfrac{-1}{1+x}\right) = -\dfrac{d}{dx}\left[\sum\limits_{n=0}^{\infty}(-1)^n x^n\right]$ [from Exercise 3]

$= \sum\limits_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}$ [from Theorem 2(i)] $= \sum\limits_{n=0}^{\infty}(-1)^n(n+1)x^n$ with $R = 1$.

In the last step, note that we *decreased* the initial value of the summation variable n by 1, and then *increased* each occurrence of n in the term by 1 [also note that $(-1)^{n+2} = (-1)^n$].

(b) $f(x) = \dfrac{1}{(1+x)^3} = -\dfrac{1}{2}\dfrac{d}{dx}\left[\dfrac{1}{(1+x)^2}\right] = -\dfrac{1}{2}\dfrac{d}{dx}\left[\sum\limits_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$ [from part (a)]

$= -\frac{1}{2}\sum\limits_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1} = \frac{1}{2}\sum\limits_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ with $R = 1$.

(c) $f(x) = \dfrac{x^2}{(1+x)^3} = x^2 \cdot \dfrac{1}{(1+x)^3} = x^2 \cdot \dfrac{1}{2}\sum\limits_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ [from part (b)]

$= \dfrac{1}{2}\sum\limits_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$

To write the power series with x^n rather than x^{n+2}, we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable by 2. This gives us $\dfrac{1}{2}\sum\limits_{n=2}^{\infty}(-1)^n(n)(n-1)x^n$ with $R = 1$.

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16. (a) $\int \frac{1}{1-x}\,dx = -\ln(1-x) + C$ and

$$\int \frac{1}{1-x}\,dx = \int (1 + x + x^2 + \cdots)\,dx = \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) + C = \sum_{n=1}^{\infty} \frac{x^n}{n} + C \text{ for } |x| < 1.$$

So $-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} + C$ and letting $x = 0$ gives $0 = C$. Thus, $f(x) = \ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ with $R = 1$.

(b) $f(x) = x\ln(1-x) = -x\sum_{n=1}^{\infty} \frac{x^n}{n} = -\sum_{n=1}^{\infty} \frac{x^{n+1}}{n}$.

(c) Letting $x = \frac{1}{2}$ gives $\ln\frac{1}{2} = -\sum_{n=1}^{\infty} \frac{(1/2)^n}{n} \quad\Rightarrow\quad \ln 1 - \ln 2 = -\sum_{n=1}^{\infty} \frac{1^n}{n2^n} \quad\Rightarrow\quad \ln 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n}$.

17. We know that $\frac{1}{1+4x} = \frac{1}{1-(-4x)} = \sum_{n=0}^{\infty} (-4x)^n$. Differentiating, we get

$$\frac{-4}{(1+4x)^2} = \sum_{n=1}^{\infty} (-4)^n n x^{n-1} = \sum_{n=0}^{\infty} (-4)^{n+1}(n+1)x^n, \text{ so}$$

$$f(x) = \frac{x}{(1+4x)^2} = \frac{-x}{4}\cdot\frac{-4}{(1+4x)^2} = \frac{-x}{4}\sum_{n=0}^{\infty} (-4)^{n+1}(n+1)x^n = \sum_{n=0}^{\infty} (-1)^n 4^n (n+1)x^{n+1}$$

for $|-4x| < 1 \quad\Leftrightarrow\quad |x| < \frac{1}{4}$, so $R = \frac{1}{4}$.

18. $\frac{1}{2-x} = \frac{1}{2(1-x/2)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}x^n$. Now $\frac{d}{dx}\left(\frac{1}{2-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{1}{2^{n+1}}x^n\right) \Rightarrow$

$$\frac{1}{(2-x)^2} = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} n x^{n-1} \text{ and } \frac{d}{dx}\left(\frac{1}{(2-x)^2}\right) = \frac{d}{dx}\left(\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} n x^{n-1}\right) \Rightarrow$$

$$\frac{2}{(2-x)^3} = \sum_{n=2}^{\infty} \frac{1}{2^{n+1}} n(n-1)x^{n-2} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^{n+3}}x^n.$$

Thus, $f(x) = \left(\frac{x}{2-x}\right)^3 = \frac{x^3}{(2-x)^3} = \frac{x^3}{2}\cdot\frac{2}{(2-x)^3} = \frac{x^3}{2}\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^{n+3}}x^n = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^{n+4}}x^{n+3}$

for $\left|\frac{x}{2}\right| < 1 \quad\Leftrightarrow\quad |x| < 2$, so $R = 2$.

19. By Example 4, $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$. Thus,

$$\begin{aligned} f(x) = \frac{1+x}{(1-x)^2} &= \frac{1}{(1-x)^2} + \frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n + \sum_{n=0}^{\infty} (n+1)x^{n+1} \\ &= \sum_{n=0}^{\infty} (n+1)x^n + \sum_{n=1}^{\infty} nx^n \qquad \text{[make the starting values equal]} \\ &= 1 + \sum_{n=1}^{\infty} [(n+1)+n]x^n = 1 + \sum_{n=1}^{\infty} (2n+1)x^n = \sum_{n=0}^{\infty} (2n+1)x^n \text{ with } R = 1. \end{aligned}$$

20. By Example 4, $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$, so

$$\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (n+1)x^n\right) \quad\Rightarrow\quad \frac{2}{(1-x)^3} = \sum_{n=1}^{\infty} (n+1)nx^{n-1}. \text{ Thus,}$$

$$f(x) = \frac{x^2+x}{(1-x)^3} = \frac{x^2}{(1-x)^3} + \frac{x}{(1-x)^3} = \frac{x^2}{2} \cdot \frac{2}{(1-x)^3} + \frac{x}{2} \cdot \frac{2}{(1-x)^3}$$

$$= \frac{x^2}{2} \sum_{n=1}^{\infty} (n+1)nx^{n-1} + \frac{x}{2} \sum_{n=1}^{\infty} (n+1)nx^{n-1} = \sum_{n=1}^{\infty} \frac{(n+1)n}{2} x^{n+1} + \sum_{n=1}^{\infty} \frac{(n+1)n}{2} x^n$$

$$= \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^n + \sum_{n=1}^{\infty} \frac{(n+1)n}{2} x^n \quad \text{[make the exponents on } x \text{ equal by changing an index]}$$

$$= \sum_{n=2}^{\infty} \frac{n^2-n}{2} x^n + x + \sum_{n=2}^{\infty} \frac{n^2+n}{2} x^n \quad \text{[make the starting values equal]}$$

$$= x + \sum_{n=2}^{\infty} n^2 x^n = \sum_{n=1}^{\infty} n^2 x^n \text{ with } R = 1.$$

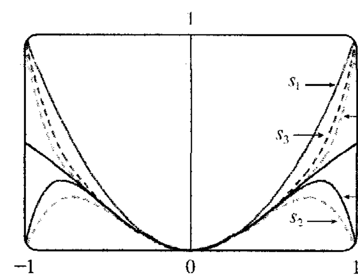
21. $f(x) = \ln(5-x) = -\int \frac{dx}{5-x} = -\frac{1}{5} \int \frac{dx}{1-x/5} = -\frac{1}{5} \int \left[\sum_{n=0}^{\infty} \left(\frac{x}{5} \right)^n \right] dx$

$$= C - \frac{1}{5} \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^n(n+1)} = C - \sum_{n=1}^{\infty} \frac{x^n}{n5^n}$$

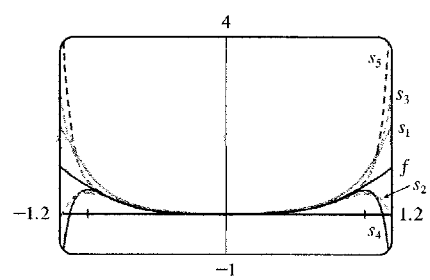
Putting $x = 0$, we get $C = \ln 5$. The series converges for $|x/5| < 1 \iff |x| < 5$, so $R = 5$.

22. $f(x) = x^2 \tan^{-1}(x^3) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{2n+1}$ [by Example 7] $= \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3+2}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+5}}{2n+1}$ for $|x^3| < 1 \iff |x| < 1$, so $R = 1$.

23. $f(x) = \frac{x^2}{x^2+1} = x^2 \left(\frac{1}{1-(-x^2)} \right) = x^2 \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n+2}$. This series converges when $|-x^2| < 1 \iff x^2 < 1 \iff |x| < 1$, so $R = 1$. The partial sums are $s_1 = x^2$, $s_2 = s_1 - x^4$, $s_3 = s_2 + x^6$, $s_4 = s_3 - x^8$, $s_5 = s_4 + x^{10}$, ... Note that s_1 corresponds to the first term of the infinite sum, regardless of the value of the summation variable and the value of the exponent. As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-1, 1)$.



24. From Example 5, we have $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ with $|x| < 1$, so $f(x) = \ln(1+x^4) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{4n}}{n}$ with $|x^4| < 1 \iff |x| < 1$ $[R = 1]$. The partial sums are $s_1 = x^4$, $s_2 = s_1 - \frac{1}{2}x^8$, $s_3 = s_2 + \frac{1}{3}x^{12}$, $s_4 = s_3 - \frac{1}{4}x^{16}$, $s_5 = s_4 + \frac{1}{5}x^{20}$, ... Note that s_1 corresponds to the first term of the infinite sum, regardless of the value of the summation variable and the value of the exponent. As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $[-1, 1]$. (When $x = \pm 1$, the series is the convergent alternating harmonic series.)



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25. $f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \int \frac{dx}{1+x} + \int \frac{dx}{1-x} = \int \frac{dx}{1-(-x)} + \int \frac{dx}{1-x}$

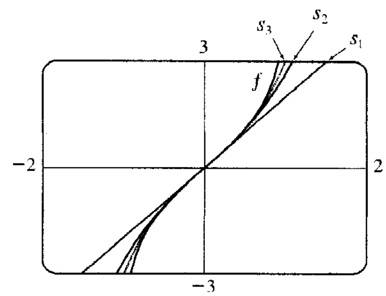
$= \int \left[\sum_{n=0}^{\infty}(-1)^n x^n + \sum_{n=0}^{\infty} x^n\right] dx = \int [(1 - x + x^2 - x^3 + x^4 - \cdots) + (1 + x + x^2 + x^3 + x^4 + \cdots)]\, dx$

$= \int (2 + 2x^2 + 2x^4 + \cdots)\, dx = \int \sum_{n=0}^{\infty} 2x^{2n}\, dx = C + \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$

But $f(0) = \ln \frac{1}{1} = 0$, so $C = 0$ and we have $f(x) = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$ with $R = 1$. If $x = \pm 1$, then $f(x) = \pm 2 \sum_{n=0}^{\infty} \frac{1}{2n+1}$, which both diverge by the Limit Comparison Test with $b_n = \frac{1}{n}$.

The partial sums are $s_1 = \frac{2x}{1}$, $s_2 = s_1 + \frac{2x^3}{3}$, $s_3 = s_2 + \frac{2x^5}{5}$, ...

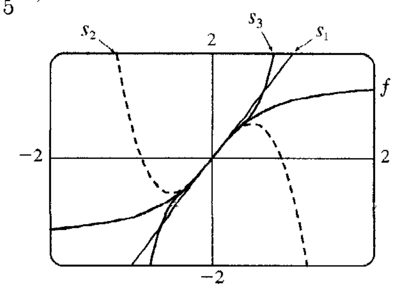
As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-1, 1)$.



26. $f(x) = \tan^{-1}(2x) = 2\int \frac{dx}{1+4x^2} = 2\int \sum_{n=0}^{\infty}(-1)^n (4x^2)^n\, dx = 2\int \sum_{n=0}^{\infty}(-1)^n 4^n x^{2n}\, dx$

$= C + 2\sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}$ $\quad [f(0) = \tan^{-1} 0 = 0$, so $C = 0]$

The series converges when $|4x^2| < 1 \Leftrightarrow |x| < \frac{1}{2}$, so $R = \frac{1}{2}$. If $x = \pm\frac{1}{2}$, then $f(x) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}$ and $f(x) = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{2n+1}$, respectively. Both series converge by the Alternating Series Test. The partial sums are $s_1 = \frac{2x}{1}$, $s_2 = s_1 - \frac{2^3x^3}{3}$, $s_3 = s_2 + \frac{2^5x^5}{5}$, ...



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

27. $\frac{t}{1-t^8} = t \cdot \frac{1}{1-t^8} = t\sum_{n=0}^{\infty}(t^8)^n = \sum_{n=0}^{\infty} t^{8n+1} \Rightarrow \int \frac{t}{1-t^8}\, dt = C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}$. The series for $\frac{1}{1-t^8}$ converges when $|t^8| < 1 \Leftrightarrow |t| < 1$, so $R = 1$ for that series and also the series for $t/(1-t^8)$. By Theorem 2, the series for $\int \frac{t}{1-t^8}\, dt$ also has $R = 1$.

28. $\dfrac{t}{1+t^3} = t \cdot \dfrac{1}{1-(-t^3)} = t \sum_{n=0}^{\infty} (-t^3)^n = \sum_{n=0}^{\infty} (-1)^n t^{3n+1} \Rightarrow \displaystyle\int \frac{t}{1+t^3}\,dt = C + \sum_{n=0}^{\infty} (-1)^n \frac{t^{3n+2}}{3n+2}$. The series for $\dfrac{1}{1+t^3}$ converges when $|-t^3| < 1 \Leftrightarrow |t| < 1$, so $R = 1$ for that series and also for the series $\dfrac{t}{1+t^3}$. By Theorem 2, the series for $\displaystyle\int \frac{t}{1+t^3}\,dt$ also has $R = 1$.

29. From Example 5, $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n}$ for $|x| < 1$, so $x^2 \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^{n+2}}{n}$ and $\displaystyle\int x^2 \ln(1+x)\,dx = C + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n+3}}{n(n+3)}$. $R = 1$ for the series for $\ln(1+x)$, so $R = 1$ for the series representing $x^2 \ln(1+x)$ as well. By Theorem 2, the series for $\displaystyle\int x^2 \ln(1+x)\,dx$ also has $R = 1$.

30. From Example 6, $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ for $|x| < 1$, so $\dfrac{\tan^{-1} x}{x} = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{2n+1}$ and $\displaystyle\int \frac{\tan^{-1} x}{x}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2}$. $R = 1$ for the series for $\tan^{-1} x$, so $R = 1$ for the series representing $\dfrac{\tan^{-1} x}{x}$ as well. By Theorem 2, the series for $\displaystyle\int \frac{\tan^{-1} x}{x}\,dx$ also has $R = 1$.

31. $\dfrac{x}{1+x^3} = x\left[\dfrac{1}{1-(-x^3)}\right] = x \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n+1} \Rightarrow$

$\displaystyle\int \frac{x}{1+x^3}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{3n+1}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{3n+2}$. Thus,

$$I = \int_0^{0.3} \frac{x}{1+x^3}\,dx = \left[\frac{x^2}{2} - \frac{x^5}{5} + \frac{x^8}{8} - \frac{x^{11}}{11} + \cdots\right]_0^{0.3} = \frac{(0.3)^2}{2} - \frac{(0.3)^5}{5} + \frac{(0.3)^8}{8} - \frac{(0.3)^{11}}{11} + \cdots.$$

The series is alternating, so if we use the first three terms, the error is at most $(0.3)^{11}/11 \approx 1.6 \times 10^{-7}$. So $I \approx (0.3)^2/2 - (0.3)^5/5 + (0.3)^8/8 \approx 0.044\,522$ to six decimal places.

32. We substitute $x/2$ for x in Example 6, and find that

$$\begin{aligned}\int \arctan\frac{x}{2}\,dx &= \int \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n+1}}{2n+1}\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^{2n+1}(2n+1)}\,dx \\ &= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2^{2n+1}(2n+1)(2n+2)}\end{aligned}$$

Thus,

$$\begin{aligned}I = \int_0^{1/2} \arctan\frac{x}{2}\,dx &= \left[\frac{x^2}{2(1)(2)} - \frac{x^4}{2^3(3)(4)} + \frac{x^6}{2^5(5)(6)} - \frac{x^8}{2^7(7)(8)} + \frac{x^{10}}{2^9(9)(10)} - \cdots\right]_0^{1/2} \\ &= \frac{1}{2^3(1)(2)} - \frac{1}{2^7(3)(4)} + \frac{1}{2^{11}(5)(6)} - \frac{1}{2^{15}(7)(8)} + \frac{1}{2^{19}(9)(10)} - \cdots\end{aligned}$$

The series is alternating, so if we use four terms, the error is at most $1/(2^{19} \cdot 90) \approx 2.1 \times 10^{-8}$. So $I \approx \dfrac{1}{16} - \dfrac{1}{1536} + \dfrac{1}{61{,}440} - \dfrac{1}{1{,}835{,}008} \approx 0.061\,865$ to six decimal places.

[continued]