HW #8, Section 9.3 Solutions

(c) From part (a), $dy/dt = -\frac{1}{50}(y - 20)$. With $t_0 = 0$, $y_0 = 95$, and $h = 2$ min, we get

$$\begin{aligned}
y_1 &= y_0 + hF(t_0, y_0) = 95 + 2\left[-\tfrac{1}{50}(95 - 20)\right] = 92\\
y_2 &= y_1 + hF(t_1, y_1) = 92 + 2\left[-\tfrac{1}{50}(92 - 20)\right] = 89.12\\
y_3 &= y_2 + hF(t_2, y_2) = 89.12 + 2\left[-\tfrac{1}{50}(89.12 - 20)\right] = 86.3552\\
y_4 &= y_3 + hF(t_3, y_3) = 86.3552 + 2\left[-\tfrac{1}{50}(86.3552 - 20)\right] = 83.700992\\
y_5 &= y_4 + hF(t_4, y_4) = 83.700992 + 2\left[-\tfrac{1}{50}(83.700992 - 20)\right] = 81.15295232
\end{aligned}$$

Thus, $y(10) \approx 81.15°\text{C}$.

9.3 Separable Equations

1. $\dfrac{dy}{dx} = 3x^2y^2 \;\Rightarrow\; \dfrac{dy}{y^2} = 3x^2\,dx \quad [y \neq 0] \;\Rightarrow\; \int y^{-2}\,dy = \int 3x^2\,dx \;\Rightarrow\; -y^{-1} = x^3 + C \;\Rightarrow$

$\dfrac{-1}{y} = x^3 + C \;\Rightarrow\; y = \dfrac{-1}{x^3 + C}$. $y = 0$ is also a solution.

2. $\dfrac{dy}{dx} = \dfrac{x}{y^4} \;\Rightarrow\; y^4\,dy = x\,dx \;\Rightarrow\; \int y^4\,dy = \int x\,dx \;\Rightarrow\; \frac{1}{5}y^5 = \frac{1}{2}x^2 + K \;\Rightarrow\; y^5 = \frac{5}{2}x^2 + 5K \;\Rightarrow$

$y = \sqrt[5]{\frac{5}{2}x^2 + C}$, where $C = 5K$.

3. $\dfrac{dy}{dx} = x\sqrt{y} \;\Rightarrow\; \dfrac{dy}{\sqrt{y}} = x\,dx \quad [y \neq 0] \;\Rightarrow\; \int y^{-1/2}\,dy = \int x\,dx \;\Rightarrow\; 2y^{1/2} = \frac{1}{2}x^2 + K \;\Rightarrow$

$\sqrt{y} = \frac{1}{4}x^2 + \frac{1}{2}K \;\Rightarrow\; y = \left(\frac{1}{4}x^2 + C\right)^2$, where $C = \frac{1}{2}K$. $y = 0$ is also a solution.

4. $xy' = y + 3 \;\Rightarrow\; x\dfrac{dy}{dx} = y + 3 \;\Rightarrow\; \dfrac{dy}{y+3} = \dfrac{dx}{x} \quad [x \neq 0, y \neq -3] \;\Rightarrow\; \displaystyle\int \frac{1}{y+3}\,dy = \int \frac{1}{x}\,dx \;\Rightarrow$

$\ln|y + 3| = \ln|x| + C \;\Rightarrow\; |y + 3| = e^{\ln|x| + C} = e^{\ln|x|}e^C = e^C|x| \;\Rightarrow\; y + 3 = kx$, where $k = \pm e^C$. Thus, $y = kx - 3$. (In our derivation, k was nonzero, but we can restore the excluded case $y = -3$ by allowing k to be zero.)

5. $xyy' = x^2 + 1 \;\Rightarrow\; xy\dfrac{dy}{dx} = x^2 + 1 \;\Rightarrow\; y\,dy = \dfrac{x^2+1}{x}\,dx \quad [x \neq 0] \;\Rightarrow\; \displaystyle\int y\,dy = \int\left(x + \frac{1}{x}\right)dx \;\Rightarrow$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + \ln|x| + K \;\Rightarrow\; y^2 = x^2 + 2\ln|x| + 2K \;\Rightarrow\; y = \pm\sqrt{x^2 + 2\ln|x| + C}$, where $C = 2K$.

6. $y' + xe^y = 0 \;\Rightarrow\; \dfrac{dy}{dx} = -xe^y \;\Rightarrow\; e^{-y}\,dy = -x\,dx \;\Rightarrow\; \int e^{-y}\,dy = \int -x\,dx \;\Rightarrow\; -e^{-y} = -\frac{1}{2}x^2 + C \;\Rightarrow$

$e^{-y} = \frac{1}{2}x^2 - C \;\Rightarrow\; -y = \ln\left(\frac{1}{2}x^2 - C\right) \;\Rightarrow\; y = -\ln\left(\frac{1}{2}x^2 - C\right)$

7. $(e^y - 1)y' = 2 + \cos x \;\Rightarrow\; (e^y - 1)\dfrac{dy}{dx} = 2 + \cos x \;\Rightarrow\; (e^y - 1)\,dy = (2 + \cos x)\,dx \;\Rightarrow$

$\int (e^y - 1)\,dy = \int (2 + \cos x)\,dx \;\Rightarrow\; e^y - y = 2x + \sin x + C$. We cannot solve explicitly for y.

8. $\dfrac{dy}{dx} = 2x(y^2+1) \;\Rightarrow\; \dfrac{dy}{y^2+1} = 2x\,dx \;\Rightarrow\; \displaystyle\int \frac{1}{y^2+1}\,dy = \int 2x\,dx \;\Rightarrow\; \tan^{-1} y = x^2 + C \;\Rightarrow$

$y = \tan(x^2 + C)$

9. $\dfrac{dp}{dt} = t^2p - p + t^2 - 1 = p(t^2-1) + 1(t^2-1) = (p+1)(t^2-1) \;\Rightarrow\; \dfrac{1}{p+1}\,dp = (t^2-1)\,dt \;\Rightarrow$

$\displaystyle\int \frac{1}{p+1}\,dp = \int (t^2-1)\,dt \;\Rightarrow\; \ln|p+1| = \tfrac{1}{3}t^3 - t + C \;\Rightarrow\; |p+1| = e^{t^3/3 - t + C} \;\Rightarrow\; p+1 = \pm e^C e^{t^3/3 - t} \;\Rightarrow$

$p = Ke^{t^3/3 - t} - 1$, where $K = \pm e^C$. Since $p = -1$ is also a solution, K can equal 0, and hence, K can be any real number.

10. $\dfrac{dz}{dt} + e^{t+z} = 0 \;\Rightarrow\; \dfrac{dz}{dt} = -e^t e^z \;\Rightarrow\; \int e^{-z}\,dz = -\int e^t\,dt \;\Rightarrow\; -e^{-z} = -e^t + C \;\Rightarrow\; e^{-z} = e^t - C \;\Rightarrow$

$\dfrac{1}{e^z} = e^t - C \;\Rightarrow\; e^z = \dfrac{1}{e^t - C} \;\Rightarrow\; z = \ln\left(\dfrac{1}{e^t - C}\right) \;\Rightarrow\; z = -\ln(e^t - C)$

11. $\dfrac{d\theta}{dt} = \dfrac{t\sec\theta}{\theta e^{t^2}} \;\Rightarrow\; \theta\cos\theta\,d\theta = te^{-t^2}\,dt \;\Rightarrow\; \int \theta\cos\theta\,d\theta = \int te^{-t^2}\,dt \;\Rightarrow$

$\theta\sin\theta + \cos\theta = -\tfrac{1}{2}e^{-t^2} + C$ [by parts]. We cannot solve explicitly for θ.

12. $\dfrac{dH}{dR} = \dfrac{RH^2\sqrt{1+R^2}}{\ln H} \;\Rightarrow\; \dfrac{\ln H}{H^2}\,dH = R\sqrt{1+R^2}\,dR \;\Rightarrow\; \displaystyle\int \frac{\ln H}{H^2}\,dH = \int R(1+R^2)^{1/2}\,dR \;\Rightarrow$

$-\dfrac{\ln H}{H} - \dfrac{1}{H} = \dfrac{1}{3}(1+R^2)^{3/2} + C$ [by parts]. We cannot solve explicitly for H.

13. $\dfrac{dy}{dx} = xe^y \;\Rightarrow\; e^{-y}\,dy = x\,dx \;\Rightarrow\; \int e^{-y}\,dy = \int x\,dx \;\Rightarrow\; -e^{-y} = \tfrac{1}{2}x^2 + C.$

$y(0) = 0 \;\Rightarrow\; -e^{-0} = \tfrac{1}{2}(0)^2 + C \;\Rightarrow\; C = -1$, so $-e^{-y} = \tfrac{1}{2}x^2 - 1 \;\Rightarrow\; e^{-y} = -\tfrac{1}{2}x^2 + 1 \;\Rightarrow$

$-y = \ln\left(1 - \tfrac{1}{2}x^2\right) \;\Rightarrow\; y = -\ln\left(1 - \tfrac{1}{2}x^2\right).$

14. $\dfrac{dP}{dt} = \sqrt{Pt} \;\Rightarrow\; dP/\sqrt{P} = \sqrt{t}\,dt \;\Rightarrow\; \int P^{-1/2}\,dP = \int t^{1/2}\,dt \;\Rightarrow\; 2P^{1/2} = \tfrac{2}{3}t^{3/2} + C.$

$P(1) = 2 \;\Rightarrow\; 2\sqrt{2} = \tfrac{2}{3} + C \;\Rightarrow\; C = 2\sqrt{2} - \tfrac{2}{3}$, so $2P^{1/2} = \tfrac{2}{3}t^{3/2} + 2\sqrt{2} - \tfrac{2}{3} \;\Rightarrow\; \sqrt{P} = \tfrac{1}{3}t^{3/2} + \sqrt{2} - \tfrac{1}{3} \;\Rightarrow$

$P = \left(\tfrac{1}{3}t^{3/2} + \sqrt{2} - \tfrac{1}{3}\right)^2.$

15. $\dfrac{dA}{dr} = Ab^2\cos br \;\Rightarrow\; \dfrac{dA}{A} = b^2\cos br\,dr \;\Rightarrow\; \displaystyle\int \frac{1}{A}\,dA = \int b^2\cos br\,dr \;\Rightarrow\; \ln|A| = b\sin br + C.$

$A(0) = b^3 \;\Rightarrow\; \ln\left|b^3\right| = b\sin 0 + C \;\Rightarrow\; C = \ln\left|b^3\right|$, so $\ln|A| = b\sin br + \ln\left|b^3\right| \;\Rightarrow$

$|A| = e^{b\sin br + \ln|b^3|} = e^{b\sin br}e^{\ln|b^3|} = \left|b^3\right|e^{b\sin br} \;\Rightarrow\; A = \pm b^3 e^{b\sin br}$. Since $A(0) = b^3$, the solution is

$A = b^3 e^{b\sin br}.$

16. $x^2y' = k\sec y \;\Rightarrow\; x^2\dfrac{dy}{dx} = \dfrac{k}{\cos y} \;\Rightarrow\; \cos y\,dy = k\,\dfrac{dx}{x^2} \;\Rightarrow\; \int \cos y\,dy = \int kx^{-2}dx \;\Rightarrow\; \sin y = -kx^{-1} + C.$

$y(1) = \tfrac{\pi}{6} \;\Rightarrow\; \sin\tfrac{\pi}{6} = -k(1)^{-1} + C \;\Rightarrow\; \tfrac{1}{2} = -k + C \;\Rightarrow\; C = \tfrac{1}{2} + k$, so

$\sin y = -\dfrac{k}{x} + \dfrac{1}{2} + k \;\Rightarrow\; y = \sin^{-1}\left(-\dfrac{k}{x} + \dfrac{1}{2} + k\right).$

17. $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u}$, $u(0) = -5$. $\int 2u\,du = \int (2t + \sec^2 t)\,dt \;\Rightarrow\; u^2 = t^2 + \tan t + C$,

where $[u(0)]^2 = 0^2 + \tan 0 + C \;\Rightarrow\; C = (-5)^2 = 25$. Therefore, $u^2 = t^2 + \tan t + 25$, so $u = \pm\sqrt{t^2 + \tan t + 25}$.

Since $u(0) = -5 < 0$, we must have $u = -\sqrt{t^2 + \tan t + 25}$.

18. $x + 3y^2\sqrt{x^2+1}\,\dfrac{dy}{dx} = 0 \;\Rightarrow\; 3y^2\sqrt{x^2+1}\,\dfrac{dy}{dx} = -x \;\Rightarrow\; 3y^2\,dy = \dfrac{-x}{\sqrt{x^2+1}}\,dx \;\Rightarrow$

$\int 3y^2\,dy = \int -x(x^2+1)^{-1/2}\,dx \;\Rightarrow\; y^3 = -(x^2+1)^{1/2} + C$. $y(0) = 1 \;\Rightarrow\; 1^3 = -(0^2+1)^{1/2} + C \;\Rightarrow$

$C = 2$, so $y^3 = -(x^2+1)^{1/2} + 2 \;\Rightarrow\; y = (2 - \sqrt{x^2+1})^{1/3}$.

19. $x \ln x = y\left(1 + \sqrt{3+y^2}\right) y'$, $y(1) = 1$. $\int x \ln x\,dx = \int \left(y + y\sqrt{3+y^2}\right)\,dy \;\Rightarrow\; \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x\,dx$

[use parts with $u = \ln x$, $dv = x dx$] $= \frac{1}{2}y^2 + \frac{1}{3}(3+y^2)^{3/2} \;\Rightarrow\; \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C = \frac{1}{2}y^2 + \frac{1}{3}(3+y^2)^{3/2}$.

Now $y(1) = 1 \;\Rightarrow\; 0 - \frac{1}{4} + C = \frac{1}{2} + \frac{1}{3}(4)^{3/2} \;\Rightarrow\; C = \frac{1}{2} + \frac{8}{3} + \frac{1}{4} = \frac{41}{12}$, so

$\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + \frac{41}{12} = \frac{1}{2}y^2 + \frac{1}{3}(3+y^2)^{3/2}$. We do not solve explicitly for y.

20. $\dfrac{dy}{dx} = \dfrac{x \sin x}{y} \;\Rightarrow\; y\,dy = x \sin x\,dx \;\Rightarrow\; \int y\,dy = \int x \sin x\,dx \;\Rightarrow\; \frac{1}{2}y^2 = -x\cos x + \sin x + C$ [by parts].

$y(0) = -1 \;\Rightarrow\; \frac{1}{2}(-1)^2 = -0\cos 0 + \sin 0 + C \;\Rightarrow\; C = \frac{1}{2}$, so $\frac{1}{2}y^2 = -x\cos x + \sin x + \frac{1}{2} \;\Rightarrow$

$y^2 = -2x\cos x + 2\sin x + 1 \;\Rightarrow\; y = -\sqrt{-2x\cos x + 2\sin x + 1}$ since $y(0) = -1 < 0$.

21. $\dfrac{dy}{dx} = \dfrac{x}{y} \;\Rightarrow\; y\,dy = x\,dx \;\Rightarrow\; \int y\,dy = \int x\,dx \;\Rightarrow\; \frac{1}{2}y^2 = \frac{1}{2}x^2 + C$. $y(0) = 2 \;\Rightarrow\; \frac{1}{2}(2)^2 = \frac{1}{2}(0)^2 + C \;\Rightarrow$

$C = 2$, so $\frac{1}{2}y^2 = \frac{1}{2}x^2 + 2 \;\Rightarrow\; y^2 = x^2 + 4 \;\Rightarrow\; y = \sqrt{x^2+4}$ since $y(0) = 2 > 0$.

22. $f'(x) = x\,f(x) - x \;\Rightarrow\; \dfrac{dy}{dx} = xy - x \;\Rightarrow\; \dfrac{dy}{dx} = x(y-1) \;\Rightarrow\; \dfrac{dy}{y-1} = x\,dx$ $[y \neq 1]$ $\Rightarrow$

$\displaystyle\int \frac{dy}{y-1} = \int x\,dx \;\Rightarrow\; \ln|y-1| = \tfrac{1}{2}x^2 + C$. $f(0) = 2 \;\Rightarrow\; \ln|2-1| = \frac{1}{2}(0)^2 + C \;\Rightarrow\; C = 0$, so

$\ln|y-1| = \frac{1}{2}x^2 \;\Rightarrow\; |y-1| = e^{x^2/2} \;\Rightarrow\; y - 1 = e^{x^2/2}$ [since $f(0) = 2$] $\Rightarrow\; y = e^{x^2/2} + 1$.

23. $u = x + y \;\Rightarrow\; \dfrac{d}{dx}(u) = \dfrac{d}{dx}(x+y) \;\Rightarrow\; \dfrac{du}{dx} = 1 + \dfrac{dy}{dx}$, but $\dfrac{dy}{dx} = x + y = u$, so $\dfrac{du}{dx} = 1 + u \;\Rightarrow$

$\dfrac{du}{1+u} = dx$ $[u \neq -1]$ $\Rightarrow\; \displaystyle\int \frac{du}{1+u} = \int dx \;\Rightarrow\; \ln|1+u| = x + C \;\Rightarrow\; |1+u| = e^{x+C} \;\Rightarrow$

$1 + u = \pm e^C e^x \;\Rightarrow\; u = \pm e^C e^x - 1 \;\Rightarrow\; x + y = \pm e^C e^x - 1 \;\Rightarrow\; y = Ke^x - x - 1$, where $K = \pm e^C \neq 0$.

If $u = -1$, then $-1 = x + y \;\Rightarrow\; y = -x - 1$, which is just $y = Ke^x - x - 1$ with $K = 0$. Thus, the general solution is $y = Ke^x - x - 1$, where $K \in \mathbb{R}$.