HW #9, SEC 9.3 SOLUTIONS

Sec. 9.3



(b) If $C_0 < r/k$, then $C_0 - r/k < 0$ and the formula for $C(t)$ shows that $C(t)$ increases and $\lim_{t\to\infty} C(t) = r/k$.

As $t$ increases, the formula for $C(t)$ shows how the role of $C_0$ steadily diminishes as that of $r/k$ increases.

**46.** (a) Use 1 billion dollars as the $x$-unit and 1 day as the $t$-unit. Initially, there is \$10 billion of old currency in circulation, so all of the \$50 million returned to the banks is old. At time $t$, the amount of new currency is $x(t)$ billion dollars, so $10 - x(t)$ billion dollars of currency is old. The fraction of circulating money that is old is $[10 - x(t)]/10$, and the amount of old currency being returned to the banks each day is $\dfrac{10 - x(t)}{10}\,0.05$ billion dollars. This amount of new currency per day is introduced into circulation, so $\dfrac{dx}{dt} = \dfrac{10-x}{10} \cdot 0.05 = 0.005(10 - x)$ billion dollars per day.

(b) $\dfrac{dx}{10-x} = 0.005\,dt \;\Rightarrow\; \dfrac{-dx}{10-x} = -0.005\,dt \;\Rightarrow\; \ln(10-x) = -0.005t + c \;\Rightarrow\; 10 - x = Ce^{-0.005t}$,

where $C = e^c \;\Rightarrow\; x(t) = 10 - Ce^{-0.005t}$. From $x(0) = 0$, we get $C = 10$, so $x(t) = 10(1 - e^{-0.005t})$.

(c) The new bills make up 90% of the circulating currency when $x(t) = 0.9 \cdot 10 = 9$ billion dollars.

$9 = 10(1 - e^{-0.005t}) \;\Rightarrow\; 0.9 = 1 - e^{-0.005t} \;\Rightarrow\; e^{-0.005t} = 0.1 \;\Rightarrow\; -0.005t = -\ln 10 \;\Rightarrow$

$t = 200 \ln 10 \approx 460.517$ days $\approx 1.26$ years.

**47.** (a) Let $y(t)$ be the amount of salt (in kg) after $t$ minutes. Then $y(0) = 15$. The amount of liquid in the tank is 1000 L at all times, so the concentration at time $t$ (in minutes) is $y(t)/1000$ kg/L and $\dfrac{dy}{dt} = -\left[\dfrac{y(t)}{1000}\,\dfrac{\text{kg}}{\text{L}}\right]\left(10\,\dfrac{\text{L}}{\text{min}}\right) = -\dfrac{y(t)}{100}\,\dfrac{\text{kg}}{\text{min}}$.

$\displaystyle\int \frac{dy}{y} = -\frac{1}{100}\int dt \;\Rightarrow\; \ln y = -\frac{t}{100} + C$, and $y(0) = 15 \;\Rightarrow\; \ln 15 = C$, so $\ln y = \ln 15 - \dfrac{t}{100}$.

It follows that $\ln\left(\dfrac{y}{15}\right) = -\dfrac{t}{100}$ and $\dfrac{y}{15} = e^{-t/100}$, so $y = 15e^{-t/100}$ kg.

(b) After 20 minutes, $y = 15e^{-20/100} = 15e^{-0.2} \approx 12.3$ kg.

**48.** Let $y(t)$ be the amount of carbon dioxide in the room after $t$ minutes. Then $y(0) = 0.0015(180) = 0.27\text{ m}^3$. The amount of air in the room is $180\text{ m}^3$ at all times, so the percentage at time $t$ (in mimutes) is $y(t)/180 \times 100$, and the change in the amount of carbon dioxide with respect to time is

$$\frac{dy}{dt} = (0.0005)\left(2\,\frac{\text{m}^3}{\text{min}}\right) - \frac{y(t)}{180}\left(2\,\frac{\text{m}^3}{\text{min}}\right) = 0.001 - \frac{y}{90} = \frac{9 - 100y}{9000}\,\frac{\text{m}^3}{\text{min}}$$

Hence, $\displaystyle\int \frac{dy}{9-100y} = \int \frac{dt}{9000}$ and $-\dfrac{1}{100}\ln|9 - 100y| = \dfrac{1}{9000}t + C$. Because $y(0) = 0.27$, we have

$-\frac{1}{100}\ln 18 = C$, so $-\frac{1}{100}\ln|9 - 100y| = \frac{1}{9000}t - \frac{1}{100}\ln 18 \;\Rightarrow\; \ln|9 - 100y| = -\frac{1}{90}t + \ln 18 \;\Rightarrow$

$\ln|9 - 100y| = \ln e^{-t/90} + \ln 18 \;\Rightarrow\; \ln|9 - 100y| = \ln(18e^{-t/90})$, and $|9 - 100y| = 18e^{-t/90}$. Since $y$ is continuous, $y(0) = 0.27$, and the right-hand side is never zero, we deduce that $9 - 100y$ is always negative. Thus, $|9 - 100y| = 100y - 9$ and we have $100y - 9 = 18e^{-t/90} \;\Rightarrow\; 100y = 9 + 18e^{-t/90} \;\Rightarrow\; y = 0.09 + 0.18e^{-t/90}$. The percentage of carbon dioxide in the room is

$$p(t) = \frac{y}{180} \times 100 = \frac{0.09 + 0.18e^{-t/90}}{180} \times 100 = (0.0005 + 0.001e^{-t/90}) \times 100 = 0.05 + 0.1e^{-t/90}$$

In the long run, we have $\lim_{t\to\infty} p(t) = 0.05 + 0.1(0) = 0.05$; that is, the amount of carbon dioxide approaches 0.05% as time goes on.

**49.** Let $y(t)$ be the amount of alcohol in the vat after $t$ minutes. Then $y(0) = 0.04(500) = 20$ gal. The amount of beer in the vat is 500 gallons at all times, so the percentage at time $t$ (in minutes) is $y(t)/500 \times 100$, and the change in the amount of alcohol with respect to time $t$ is $\frac{dy}{dt}$ = rate in − rate out $= 0.06\left(5\ \frac{\text{gal}}{\text{min}}\right) - \frac{y(t)}{500}\left(5\ \frac{\text{gal}}{\text{min}}\right) = 0.3 - \frac{y}{100} = \frac{30-y}{100}\,\frac{\text{gal}}{\text{min}}$.

Hence, $\int \frac{dy}{30-y} = \int \frac{dt}{100}$ and $-\ln|30-y| = \frac{1}{100}t + C$. Because $y(0) = 20$, we have $-\ln 10 = C$, so $-\ln|30-y| = \frac{1}{100}t - \ln 10 \;\Rightarrow\; \ln|30-y| = -t/100 + \ln 10 \;\Rightarrow\; \ln|30-y| = \ln e^{-t/100} + \ln 10 \;\Rightarrow\; \ln|30-y| = \ln(10e^{-t/100}) \;\Rightarrow\; |30-y| = 10e^{-t/100}$. Since $y$ is continuous, $y(0) = 20$, and the right-hand side is never zero, we deduce that $30 - y$ is always positive. Thus, $30 - y = 10e^{-t/100} \;\Rightarrow\; y = 30 - 10e^{-t/100}$. The percentage of alcohol is $p(t) = y(t)/500 \times 100 = y(t)/5 = 6 - 2e^{-t/100}$. The percentage of alcohol after one hour is $p(60) = 6 - 2e^{-60/100} \approx 4.9$.

**50.** (a) If $y(t)$ is the amount of salt (in kg) after $t$ minutes, then $y(0) = 0$ and the total amount of liquid in the tank remains constant at 1000 L.

$$\frac{dy}{dt} = \left(0.05\ \frac{\text{kg}}{\text{L}}\right)\left(5\ \frac{\text{L}}{\text{min}}\right) + \left(0.04\ \frac{\text{kg}}{\text{L}}\right)\left(10\ \frac{\text{L}}{\text{min}}\right) - \left(\frac{y(t)}{1000}\ \frac{\text{kg}}{\text{L}}\right)\left(15\ \frac{\text{L}}{\text{min}}\right)$$

$$= 0.25 + 0.40 - 0.015y = 0.65 - 0.015y = \frac{130-3y}{200}\ \frac{\text{kg}}{\text{min}}$$

Hence, $\int \frac{dy}{130-3y} = \int \frac{dt}{200}$ and $-\frac{1}{3}\ln|130-3y| = \frac{1}{200}t + C$. Because $y(0) = 0$, we have $-\frac{1}{3}\ln 130 = C$, so $-\frac{1}{3}\ln|130-3y| = \frac{1}{200}t - \frac{1}{3}\ln 130 \;\Rightarrow\; \ln|130-3y| = -\frac{3}{200}t + \ln 130 = \ln(130e^{-3t/200})$, and $|130-3y| = 130e^{-3t/200}$. Since $y$ is continuous, $y(0) = 0$, and the right-hand side is never zero, we deduce that $130 - 3y$ is always positive. Thus, $130 - 3y = 130e^{-3t/200}$ and $y = \frac{130}{3}(1 - e^{-3t/200})$ kg.

(b) After one hour, $y = \frac{130}{3}(1 - e^{-3\cdot 60/200}) = \frac{130}{3}(1 - e^{-0.9}) \approx 25.7$ kg.

*Note:* As $t \to \infty$, $y(t) \to \frac{130}{3} = 43\frac{1}{3}$ kg.

**51.** Assume that the raindrop begins at rest, so that $v(0) = 0$. $dm/dt = km$ and $(mv)' = gm \;\Rightarrow\; mv' + vm' = gm \;\Rightarrow\; mv' + v(km) = gm \;\Rightarrow\; v' + vk = g \;\Rightarrow\; \frac{dv}{dt} = g - kv \;\Rightarrow\; \int \frac{dv}{g-kv} = \int dt \;\Rightarrow\; -(1/k)\ln|g-kv| = t + C \;\Rightarrow\; \ln|g-kv| = -kt - kC \;\Rightarrow\; g - kv = Ae^{-kt}$. $v(0) = 0 \;\Rightarrow\; A = g$. So $kv = g - ge^{-kt} \;\Rightarrow\; v = (g/k)(1 - e^{-kt})$. Since $k > 0$, as $t \to \infty$, $e^{-kt} \to 0$ and therefore, $\lim_{t\to\infty} v(t) = g/k$.