The Vector Equation of Line $L$

OP = $\vec{r} = <x, y, z>$ is a general point on Line $L$

$P_0 (x_0, y_0, z_0)$
$P_1 (x_1, y_1, z_1)$

$\vec{r}_0 = <x_0, y_0, z_0>$
$\vec{r}_1 = <x_1, y_1, z_1>$

$\vec{p}_p = \vec{r}_1 - \vec{r}_0 = \vec{v} = <a, b, c>$

$\vec{v}$ (or any $k\vec{v}, k \neq 0$) is a direction vector for $L$

The Equations of Line $L$

The Vector Equation of Line $L$ \[ \vec{OP} = \vec{r} = \vec{r}_0 + t\vec{v}, \quad -\infty < t < \infty \]

\[ <x, y, z> = <x_0, y_0, z_0> + t <a, b, c> \]

\[ <x, y, z> = <x_0 + ta, y_0 + tb, z_0 + tc> \]

The parametric equations of Line $L$.

The line segment from $\vec{r}_0$ to $\vec{r}_1$ (along $L$), $\vec{p}_p$ above, has equation: \[ \vec{r} = (1-t)\vec{r}_0 + t\vec{r}_1, \quad 0 \leq t \leq 1. \]
To review: When line \( L \) has point \( P_0 = <x_0, y_0, z_0> \) on the line and it has direction vector \( \vec{v} = <a, b, c> \), its equations are:

**Vector Equation:** \( \vec{r} = \vec{r}_0 + t \vec{v}, \ -\infty < t < \infty \)

**Parametric Equations:**
\[
\begin{align*}
    x &= x_0 + at \\
    y &= y_0 + bt \\
    z &= z_0 + ct
\end{align*}
\]

**Symmetric Equations:**
\[
\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \quad (=t)
\]

**Fact:** If lines \( L_1 \) and \( L_2 \) have parallel direction vectors, then \( L_1 \) and \( L_2 \) are parallel lines or \( L_1 = L_2 \).

**Fact:** If lines \( L_1 \) and \( L_2 \) intersect and have orthogonal direction vectors, then \( L_1 \perp L_2 \).

**Fact:** If line \( L \) is parallel to the \( yz \)-plane, then the direction vector is \( \vec{v} = <0, b, c> \) and the symmetric equations are
\[
\begin{align*}
    x &= x_0 \\
    \frac{y-y_0}{b} &= \frac{z-z_0}{c}
\end{align*}
\]
(assuming \( b \neq 0 \) and \( c \neq 0 \)).

Similarly, when \( L \) is parallel to the \( xz \)-plane or to the \( xy \)-plane.
**Problem:** Line \( L \) has direction vector \( \vec{v} = \langle 9, 5, -3 \rangle \) and it passes through the point \( P(4, -14, 5) \).

Find the vector equation of \( L \), the parametric equations of \( L \), and the symmetric equations of \( L \).

**Solution:** Direction vector \( \vec{v} = \langle 9, 5, -3 \rangle \)

Point \( \vec{r}_0 = \langle 4, -14, 5 \rangle \)

**Vector Equation:**
\[
\vec{r} = \vec{r}_0 + t \vec{v} \\
\vec{r} = \langle 4, -14, 5 \rangle + t \langle 9, 5, -3 \rangle
\]

**Parametric Equations:**
\[
\langle x, y, z \rangle = \langle 4 + 9t, -14 + 5t, 5 - 3t \rangle
\]

\[
\begin{align*}
x &= 4 + 9t \\
y &= -14 + 5t \\
z &= 5 - 3t
\end{align*}
\]

\[
\begin{align*}
9t &= x - 4 \\
5t &= y + 14 \\
-3t &= z - 5
\end{align*}
\]

Write all as \( t = \) ______

**Symmetric Equations:**
\[
\begin{align*}
\frac{x - 4}{9} &= \frac{y + 14}{5} = \frac{z - 5}{-3} \\
(= t)
\end{align*}
\]

What point on line \( L \) has \( x \)-coord = 22?

Set \( x = 22 \) in the symmetric EQs and solve for \( y \) and \( z \).

\[
\begin{align*}
\frac{22 - 4}{9} &= \frac{18}{9} = \frac{2}{1} \\
2 &= \frac{y + 14}{5} \Rightarrow 10 = y + 14 \Rightarrow y = -4 \\
2 &= \frac{z - 5}{-3} \Rightarrow -6 = z - 5 \Rightarrow z = -1
\end{align*}
\]

Point \( A(22, -4, -1) \) is on line \( L \) (use \( t = 2 \) in the parametric equations).
**Fact**: If $P_0$ and $P_1$ are two points on line $L$, then the displacement vector $\vec{w} = \overrightarrow{P_0P_1}$ is a direction vector for line $L$.

**Problem**: Line $L$ passes through points $P(2, -1, 8)$ and $Q(5, 6, -3)$.

Find the parametric equations for $L$.

**Solution**: $P(2, -1, 8)$ has position vector $\vec{r}_0 = \langle 2, -1, 8 \rangle$.

$Q(5, 6, -3)$ has position vector $\vec{r}_1 = \langle 5, 6, -3 \rangle$.

$\vec{w} = \overrightarrow{PQ} = \vec{r}_1 - \vec{r}_0 = \langle 5 - 2, 6 - (-1), -3 - 8 \rangle$

$\vec{w} = \langle 3, 7, -11 \rangle$ is a direction vector for $L$.

$\langle x, y, z \rangle = \langle 2, -1, 8 \rangle + t \langle 3, 7, -11 \rangle$

$\begin{align*}
x &= 2 + 3t \\
y &= -1 + 7t \\
z &= 8 - 11t
\end{align*}$

**Problem**: Find the vector equation and the parametric equations of the line segment from $(2, 4, -3)$ to $(3, 1, 1)$.

**Solution**: Position vector $\vec{r}_0 = \langle 2, 4, -3 \rangle$ represents $(2, 4, -3)$

Position vector $\vec{r}_1 = \langle 3, 1, 1 \rangle$ represents $(3, 1, 1)$

**Vector Equation**: $\vec{r} = (1-t)\vec{r}_0 + t\vec{r}_1 \Rightarrow \vec{r} = (1-t) \langle 2, 4, -3 \rangle$

$+ t \langle 3, 1, 1 \rangle$

$0 \leq t \leq 1$
Problem: Find the equations of line \( L_1 \) which passes through point \( P_0 (1, 6, 2) \) and which is parallel to the line \( L_2 \), where line \( L_2 \) passes through \( P_1 (5, 4, 8) \) and \( P_2 (9, 4, 0) \).

Solution: Since \( L_1 \parallel L_2 \), the direction vector for \( L_2 \) can also serve as a direction vector for \( L_1 \). The vector \( \overrightarrow{P_1P_2} = \langle 9-5, 4-4, 0-8 \rangle = \langle 4, 0, -8 \rangle \) can serve as a direction vector for \( L_2 \), but \( \overrightarrow{v} = \frac{1}{4} \langle 4, 0, -8 \rangle = \langle 1, 0, -2 \rangle \) can also be a direction vector for \( L_2 \) and so also for \( L_1 \).

The vector equation of \( L_1 \) is \( \overrightarrow{r} = \langle 1, 6, 2 \rangle + t \langle 1, 0, -2 \rangle \)

\[ \langle x, y, z \rangle = \langle 1 + t, 6 + 0t, 2 + (-2)t \rangle \]

- \( x = 1 + t \)
- \( y = 6 \)
- \( z = 2 - 2t \)

Parametric equations: \( L_1 \)

\[ t = x - 1 \]

\[ z - 2 = -2t \]

\[ t = \frac{z - 2}{-2} \]

Symmetric equations: \( x - 1 = \frac{z - 2}{-2} \), \( y = 6 \), \( z = 2 \)

Note: \( L_1 \) is parallel to the \( xz \)-plane.