The maximal degeneration toolkit, generalized theta functions and mirror symmetry

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What is it about?

We construct and study a class of canonical degenerations of algebraic varieties

$$\pi : \mathfrak{X} \to T.$$

Example. A degeneration of quartic surfaces

$$Z_0 Z_1 Z_2 Z_3 + t(Z_0^4 + Z_1^4 + Z_2^4 + Z_3^4) = 0$$

in $\mathbb{P}^3 \times T,\ T = \operatorname{Spec} \Bbbk[\![t]\!]$, $\Bbbk$ a field.

- Degeneration: π is flat and $\mathfrak{X}$ is algebraically complete (proper over an affine scheme).
- Parameter space T: The spectrum of a (complete) local ring.
- Central fibre $X_0 \subset \mathfrak{X}$: A union of toric varieties.
- Generic fibre $X_\eta \subset \mathfrak{X}$: A variety with effective anticanonical class (Calabi-Yau, Fano, affine,...)

Main features of the construction

- The construction starts from completely discrete *(tropical)* data $(B, \mathscr{P}, \varphi)$,
- and it works order by order (around the closed point $O \in T$).
- B is a real affine manifold (coordinate changes in $\mathrm{Aff}(\mathbb{Z}^n) = \mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$);
  $\mathscr{P}$ is a polyhedral decomposition of B;
  φ is a convex (multivalued) piecewise linear function on B.
- The central fibre X_0 (i.e. order 0) can be readily read off from $(B, \mathscr{P})$.
- φ provides standard toric models for the deformation $X_0 \subset \mathfrak{X}$, which however do not patch consistently.
- In each step corrections to the standard toric models of $\mathfrak{X}$ are inserted. These corrections are carried by walls, codimension one polyhedral subsets of B.
- The correction at next order is determined by a scattering procedure at transverse intersections of walls.
- The whole process only depends on the choice of initial walls, for which there is a finite-dimensional universal choice, leading to the universal base space T.

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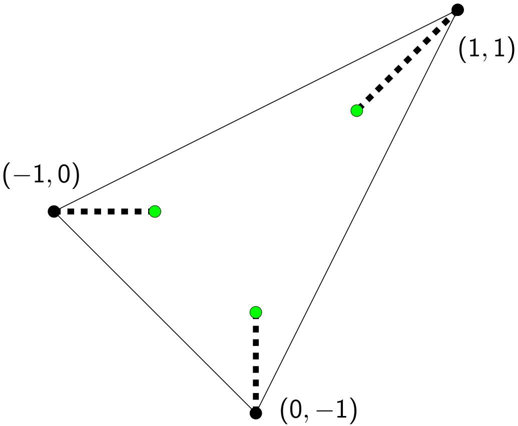
Example: A degeneration of cubic surfaces



The affine manifold B with three singular points (green).
Drawn is a fundamental domain covering B minus three line segments (dashed).

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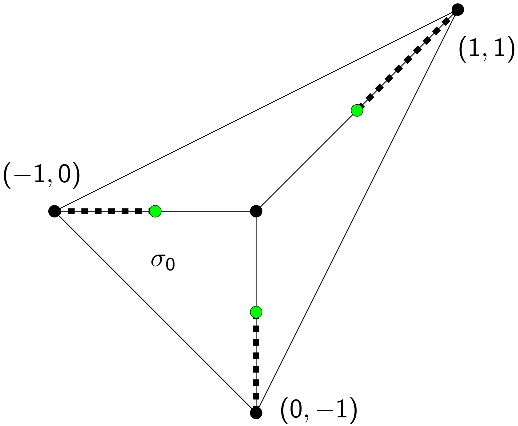

The polyhedral decomposition $\mathscr{P}$ of B with three maximal cells (triangles).
PL-function: $\varphi|_{\sigma_0} = 0$, $\varphi(1,1) = 1$.

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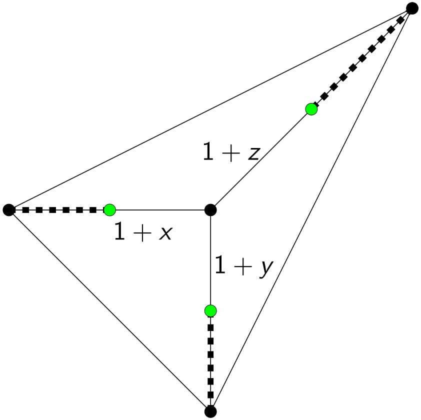

The starting data for the scattering algorithm (initial walls).

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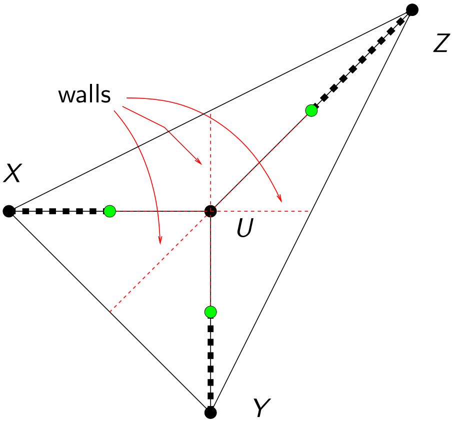

In this example three more walls suffice for all orders.

Resulting degeneration: $XYZ = t\big((1+t)U^3 + (X+Y+Z)U^2\big)$ in $\mathbb{P}^3 \times \mathbb{A}^1$.

Main features of the construction (recap)

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What is it good for? (I)

Mirror symmetry is induced by a perfect duality on discrete data:

Discrete Legendre transform: $(B, \mathscr{P}, \varphi) \quad \stackrel{1:1}{\longleftrightarrow} \quad (\check{B}, \check{\mathscr{P}}, \check{\varphi})$.

$\mathscr{P}$, $\check{\mathscr{P}}$ are dual cell decompositions.

A maximal cell $\check{\sigma} \subset \check{B}$ is the Newton polyhedron (vertices provided by slopes) of the MPL-function φ at the vertex $v \in B$ dual to $\check{\sigma}$, and vice versa.

This duality explains most of the known mirror constructions, most notably the *Batyrev-Borisov* mirror duality of complete intersections in toric varieties and the generalization by *Givental-Hori-Vafa* to semi-positive cases.

While modifications are necessary in some cases we believe that our construction together with discrete Legendre duality is the basic mechanism behind mirror symmetry.

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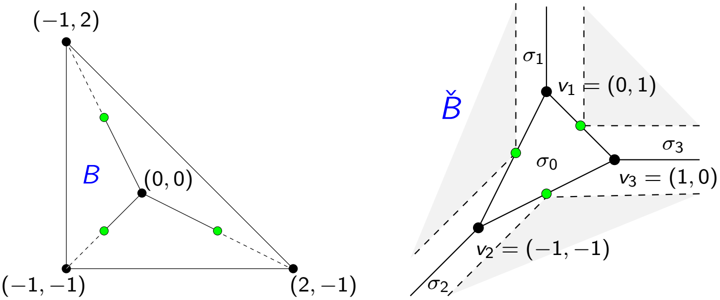
Example: Mirror symmetry for $\mathbb{P}^2$



A non-toric model of $\mathbb{P}^2$ (left) and its Legendre dual (right).

Note: $\partial B \longleftrightarrow$ anticanonical divisor (here: an elliptic curve)

$\check{B}$ unbounded $\longleftrightarrow$ $\check{\mathfrak{X}}$ comes with a canonical set of global functions, here providing the Landau-Ginzburg potential (restricts to the Givental-Hori-Vafa mirror $W = x + y + \frac{1}{xy}$ on some chart).

What is it good for? (II)

While the original motivation for our construction comes from mirror symmetry, it stands by itself and as such has many interesting properties.

Slogan: Any entity arising in mirror symmetry can be conveniently and rigorously expressed in terms of our data!

Examples[1]**:**

- (Limiting mixed) Hodge structure
- Variation of Hodge structures (period integrals).
- Canonical coordinates on the parameter space.
- Gromov-Witten invariants.
- Fukaya-category (via a tropical Morse category).
- The bounded derived category of coherent sheaves.
- Donaldson-Thomas invariants (?)
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[1] many conjectural in full generality

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What is it good for? (III)

Many other (partly conjectural) applications and relations to other work:

- Construction of interesting families of algebraic varieties with full control of their properties, e.g. of their topology.
- Generalized theta functions.
- Geometric quantization.
- A conjecture of Looijenga on the duality of certain surface singularities.
- Cluster varieties (G., Hacking, Keel, Kontsevich).
- Floer-theoretic mirror construction (Fukaya, Auroux, Abouzaid,. . .).
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Results I. Hodge theory

Fix $(B, \mathscr{P}, \varphi)$ and $\pi : \mathfrak{X} \to T$ the corresponding degeneration.

Let $i : B_0 \hookrightarrow B$ be the inclusion of the complement of the singular locus (in $\mathrm{codim}_{\mathbb{R}} = 2$) of the affine structure.

Λ_{B_0} the local system of integral vector fields on B_0.

Theorem (G./S. 2008)

With suitable hypotheses on the singularities of B,

- $H^q(X_0, \Omega^p_{X_0/\operatorname{Spec}\Bbbk^\dagger}) \cong H^q(B, i_*\bigwedge^p \Lambda_{B_0} \otimes_{\mathbb{Z}} \mathbb{C})$.
- $R^q\pi_*\Omega^p_{\mathfrak{X}/T}$ *is locally free.*
- *If $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ and $\check{\mathfrak{X}} \to \check{T}$ is the mirror-dual data then $\Lambda_{\check{B}_0} \cong \Lambda^*_{B_0}$ and*[a]

$$H^q\left(B, i_*\bigwedge^p \Lambda_{B_0} \otimes \mathbb{C}\right) \cong H^q\left(\check{B}, \check{i}_*\bigwedge^{\dim B - p} \Lambda_{\check{B}_0} \otimes \mathbb{C}\right)$$

gives the classic exchange of Hodge numbers in mirror symmetry for $\mathfrak{X} \to T$ and $\check{\mathfrak{X}} \to \check{T}$.

[a] assuming B oriented.

Results II. Canonical coordinates

For the *maximal* degenerations relevant in mirror symmetry certain period integrals over the holomorphic n-form Ω (CY case) provide distinguished holomorphic coordinates on the parameter space.[2]

$$q_\mu = \exp\left(-2\pi i \frac{\int_{\beta_\mu}\Omega}{\int_\alpha \Omega}\right), \quad \mu = 1, \ldots, r.$$

($\alpha \in W_0 \subset H_n(X_t, \mathbb{Z})$ the monodromy-invariant cycle, $\beta_\mu \in H_n(X_t, \mathbb{Z})$ a basis for W_2/W_0 of the monodromy weight filtration, for some $t \neq 0$.)

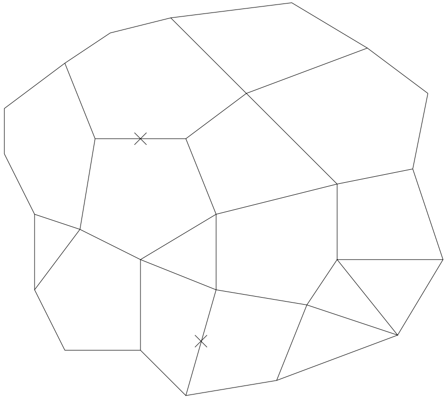
Theorem (Ruddat/S.; 2008, 2014)

The monomials of the parameter space $T = \operatorname{Spec} \Bbbk[\![Q]\!]$ are canonical coordinates.

[2] Candelas/de la Ossa/Green/Parkes, Morrison, Deligne

Results III. Scattering and Gromov-Witten theory

The inductive step of our construction inserts new walls based at intersections of previous walls, by a computation in a pro-unipotent algebraic group, introduced by Kontsevich and Soibelman in two dimensions.



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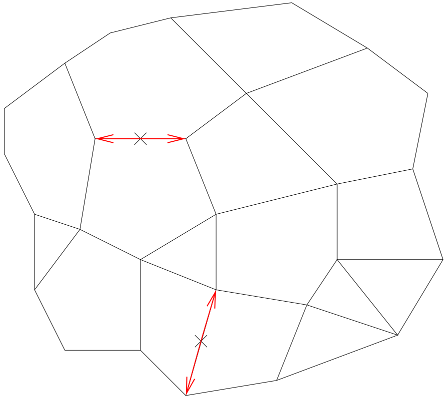

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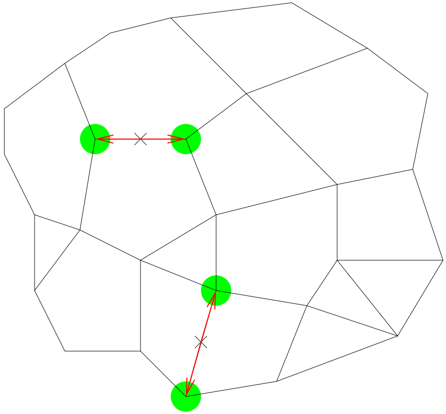

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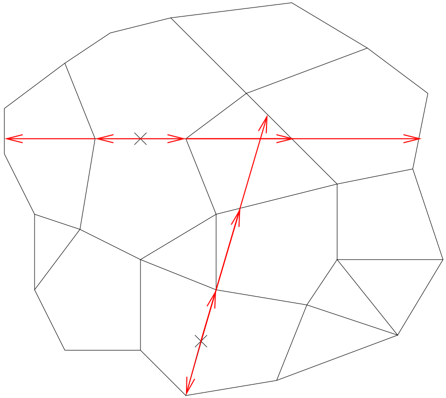

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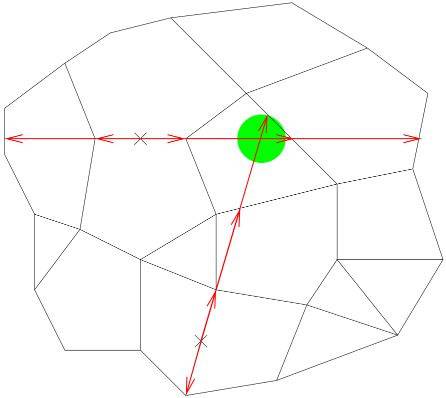

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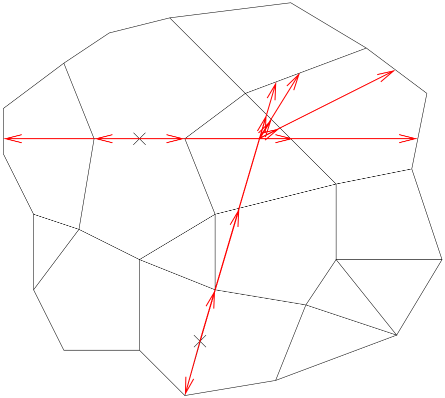


Results III. Scattering and Gromov-Witten theory

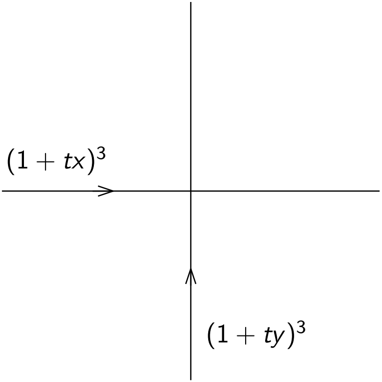
The insertion of new walls at the codimension two intersection locus of previously constructed walls takes place in the two-dimensional normal space. The two-dimensional case, in turn, is governed by holomorphic curve counting on toric surfaces:

Theorem (G./Pandharipande/S. 2009)

Given finitely many incoming walls from directions $m_1, \ldots, m_r \in \mathbb{Z}^2$ with functions $f_1, \ldots, f_r$ then the coefficients of the function $\log f_0$ on the outgoing wall in direction m_0 of the scattering diagram are given by relative genus 0 Gromov-Witten invariants of the complete toric surface with fan generated by $\mathbb{R}_{\geq 0} m_0, \ldots, \mathbb{R}_{\geq 0} m_r$.

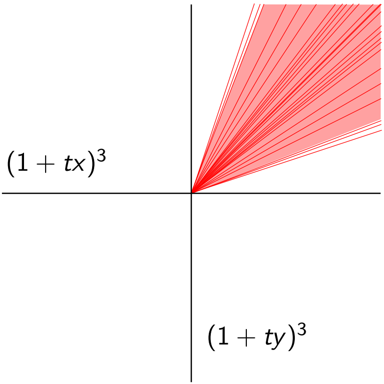
This fact is a strong manifestation of the mirror-symmetric nature of our construction.

Results III. Scattering and Gromov-Witten theory



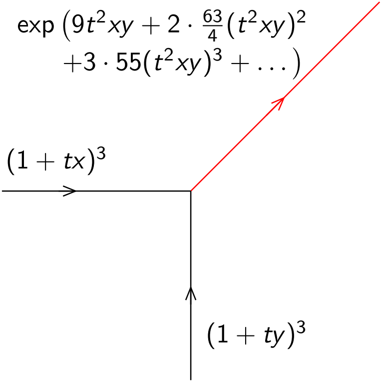
Starting point for scattering:
Intersection of two walls.

Results III. Scattering and Gromov-Witten theory



Result of scattering. The shaded area contains a dense set of walls.

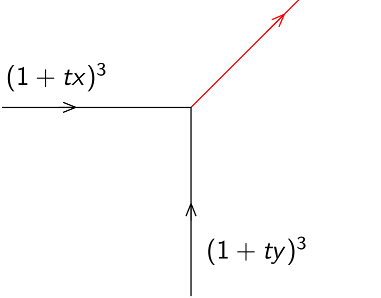
Results III. Scattering and Gromov-Witten theory

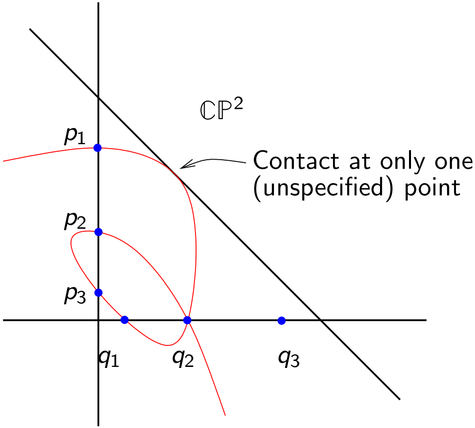


Part of scattering result with associated functions and just one outgoing wall shown

Results III. Scattering and Gromov-Witten theory

$$\exp\left(9t^2xy + 2\cdot\frac{63}{4}(t^2xy)^2 + 3\cdot 55(t^2xy)^3 + \dots\right)$$





Interpretation in terms of genus 0 curve counting on the associated toric surface (here: $\mathbb{CP}^2$): Given intersection multiplicities at fixed points p_i, q_j.
Result: $55 = 1 + 18 + 12\cdot 3$, the depicted curve contributes 3.

Results IV. Generalized theta functions

Theorem (G./Hacking/Keel/S. 2008–2014)

Our degenerations $\mathfrak{X} \to T = \operatorname{Spec} R$ come with a relatively ample line bundle $\mathfrak{L}$ and a canonical basis of sections of $\Gamma(\mathfrak{X}, \mathfrak{L}^{\otimes d})$, $d \geq 0$, as an R-module. The level $d > 0$ basis of sections ϑ_m is labelled by $m \in B\left(\frac{1}{d}\mathbb{Z}\right) \subset B$, the $1/d$-integral points of B.

The sections ϑ_m have a conjectural symplectic interpretation as counting holomorphic discs of Maslov index $2d$ on the total space of $\check{\mathfrak{L}}^{-1}$ on the mirror side.

The algebraic-geometric nature of the ϑ_m is unkown to date, but there is a possible interpretation in terms of geometric quantization.

Examples:

- For degenerations of abelian varieties we retrieve the classical theta functions.
- For degenerations of toric varieties the ϑ_m are the monomial sections.
- For the Dwork family of quintics $Z_0Z_1Z_2Z_3Z_4 + t(Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5)$ the level one ϑ_m are the linear coordinates $Z_0, \ldots, Z_4$.

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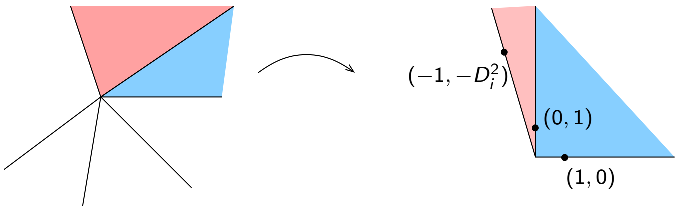
Results V. Mirrors to surfaces

Gross-Hacking-Keel (2011): (Y, D) a pair with

- Y a projective rational surface.
- $D \in |-K_Y|$ an effective divisor which is a cycle of rational curves, $D = D_1 + \cdots + D_n$.

Associate to (Y, D) a "generalized fan" (B, Σ). Here B is homeomorphic to $\mathbb{R}^2$ and has an affine structure with singularity at the origin. Σ is a fan in B, a collection of cones with vertex the singular point.



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Using relative Gromov-Witten invariants of (Y, D), specifically counts of rational curves in Y meeting D at one point, we construct a wall structure which gives an affine mirror family $\mathcal{X} \to \operatorname{Spec} \Bbbk[\![\mathrm{NE}(Y)]\!]$. Here $\mathrm{NE}(Y)$ is the monoid of effective curves, and the completion is at the zero dimensional stratum.

Features:

- The family extends over $\operatorname{Spec} \Bbbk[\mathrm{NE}(Y)]$ if D supports an ample divisor.
- If D is contractible (necessarily to a cusp singularity), the mirror family contains a smoothing of the dual cusp singularity, proving a 1981 *conjecture of Looijenga* characterizing smoothable cusp singularities.

The difference to our original construction is that the affine structure now has a singularity at a vertex. In this situation walls can not be produced inductively by scattering. The wall structure rather comes in an a priori fashion from Gromov-Witten theory of (Y, D), reversing [GPS09].

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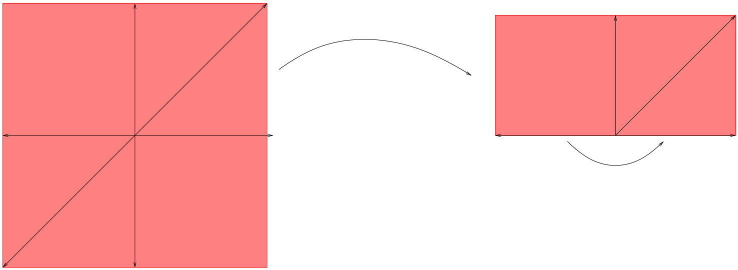
Example: Mirrors to surfaces

Singularities may go at vertices: divide $\mathbb{R}^2$ with the fan for a del Pezzo surface of degree 6 out by negation.



Example: Mirrors to surfaces (cont'd)

With no corrections, yields a degeneration of the affine Cayley cubic, $(\mathbb{C}^*)^2/(z_1, z_2) \mapsto (z_1^{-1}, z_2^{-1})$:

$$XYZ - t(X^2 + Y^2 + Z^2) + 4t^3 = 0.$$

There is a consistent wall structure coming from counts of rational curves on a cubic surface, with a ray of every slope appearing. Nevertheless, an explicit algebraic equation can be written down, and one obtains a family of in general smooth cubics. (Gross, Hacking, Keel.)

Results VI. Compactification of K3 moduli spaces

(With Hacking/Keel; in progress)

Let $\mathscr{F}_g$ be the moduli space (stack) of K3 surfaces polarized by a degree $2g-2$ line bundle. It has been a long standing problem to find a compactification $\overline{\mathscr{F}}_g$ of $\mathscr{F}_g$ that comes with a family of K3 surfaces (necessarily with certain singularities over $\overline{\mathscr{F}}_g \setminus \mathscr{F}_g$).

Morrison suggested finding $\overline{\mathscr{F}}_g$ as a toroidal compactification coming from the Mori fan of a certain one-dimensional family $\mathcal{Y} \to D$ of lattice-polarized K3-surfaces.

Using Gromov-Witten theory on the components of the (normal crossings) central fibre $Y_0 \subset \mathcal{Y}$, combined with the scattering algorithm, we can indeed produce a wall structure and hence a local partial compactification of $\mathscr{F}_g$ near each 0-dimensional stratum of certain toroidal compactifications. We believe that these can be patched together to give a global family over certain toroidal compactifications $\overline{\mathscr{F}}_g$ of $\mathscr{F}_g$.

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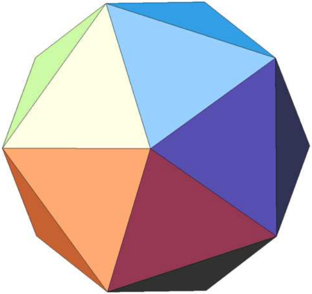
Example: A degeneration of K3-surfaces



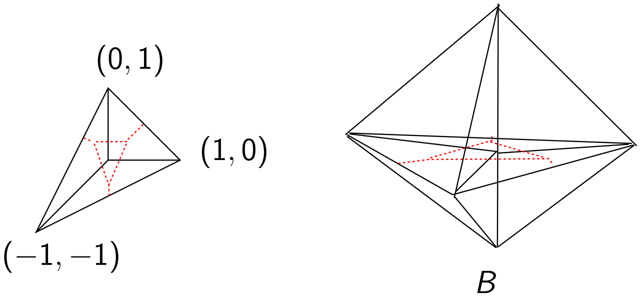
3

An icosahedron can be given an affine manifold structure with singularities at the vertices, by flattening out the edges. With a complicated wall structure, this gives a family whose central fibre X_0 is an icosahedron formed out of copies of $\mathbb{P}^2$.

This is mirror to a degenerating family of K3 surfaces constructed by Jan Stevens, with central fibre a dodecahedron of del Pezzo surfaces of degree 5.

[3] Figure from Tom Ruen (en.wikipedia)

Final example: Local mirror symmetry



This gives a (compactification) of the mirror of local $\mathbb{P}^2$. However, our construction automatically gives the canonical coordinate: rather than producing the standard expression for the mirror,

$$V(uv - (1 + x + y + tx^{-1}y^{-1})) \subset \mathbb{A}^2_{u,v} \times (\mathbb{G}^2_m)_{x,y} \times \operatorname{Spec} \Bbbk[\![t]\!],$$

it gives the equation

$$uv = 1 + x + y + tx^{-1}y^{-1} - 2t + 5t^2 - 32t^3 + 286t^4 - 3038t^5 \cdots .$$