Stochastic Perron's Method in Linear and Nonlinear Problems

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based on joint work with

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Brown University, March 5th, 2013

Outline

Objective

Prove that the value function is the unique viscosity solution of the Hamilton-Jacobi-Bellman-(Isaacs) equation, **avoiding the proof of the Dynamic Programming Principle (DPP)**.

Summary

New look at an old (set of) problem(s).

Disclaimer:

- not trying to "reinvent the wheel" but provide a different view (and a new tool)

Questions:

- why a new look?
- how?/the tool we propose

Stochastic Control Problems

State equation

$$\begin{cases} dX_t = b(t, X_t, \alpha_t)dt + \sigma(t, X_t, \alpha_t)dW_t \\ X_s = x. \end{cases}$$

$X \in \mathbb{R}^n, W \in \mathbb{R}^d$

Cost functional $J(s, x, \alpha) = \mathbb{E}[\int_s^T R(t, X_t, \alpha_t)dt + g(X_T)]$

Value function

$$v(s, x) = \sup_{\alpha} J(s, x, \alpha).$$

Comments: all formal, no filtration, admissibility, etc. Also, we have in mind other classes of control problems as well.

(My understanding of) Continuous-time DP and HJB's

Two possible approaches

1. analytic (direct)
2. probabilistic (study the properties of the value function)

The Analytic approach

1. write down the DPE/HJB

$$\begin{cases} u_t + \sup_\alpha \left\{ L_t^\alpha u + R(t,x,\alpha) \right\} = 0 \\ u(T,x) = g(x) \end{cases}$$

2. solve it i.e.
   - prove existence of a smooth solution u
   - (if lucky) find a closed form solution u
3. go over verification arguments
   - proving existence of a solution to the closed-loop SDE
   - use Itô's lemma and uniform integrability, to conclude $u = v$ and the solution of the closed-loop eq. is optimal

Analytic approach cont'd

Conclusions: the existence of a smooth solution of the HJB (with some properties) implies

1. $u = v$ (uniqueness of the smooth solution)
2. (DPP)

$$v(s, x) = \sup_{\alpha} \mathbb{E}[\int_s^{\tau} R(t, X_t, \alpha_t) dt + v(\tau, X_\tau)]$$

3. $\alpha(t, x) = \arg\max$ is the optimal feedback

Complete description: Fleming and Rishel

smooth sol of (DPE) $\rightarrow$ (DPP)+value fct is the unique sol

Probabilistic/Viscosity Approach

1. prove the (DPP)
2. show that (DPP) $\longrightarrow$ v is a viscosity solution
3. IF viscosity comparison holds, then v is the unique viscosity solution

(DPP)+visc. comparison $\rightarrow$ v is the unique visc sol(DPE)

Meta-Theorem If the value function is the unique viscosity solution, then finite difference schemes approximate the value function and the optimal feedback control (approximate backward induction works).

Comments on probabilistic approach

1. quite hard (actually very hard compared to deterministic case)
    1.1 by approx with discrete-time or smooth problems (Krylov)
    1.2 work directly on the value function (El Karoui, Borkar, Hausmann, Bouchard-Touzi for a weak version)
2. non-trivial, but easier than 1: Fleming-Soner, Bouchard-Touzi
3. has to be proved separately (analytically) anyway

Probabilistic/Viscosity Approach pushed further

Sometimes we are lucky:

- using the specific structure of the HJB can prove that a viscosity solution of the DPE is actually smooth!
- if that works we can just come back to the Analytic approach and go over step 3, i.e. we can perform verification using the smooth solution v (the value function) to obtain
  1. the (DPP)
  2. Optimal feedback control $\alpha(t, x)$

(DPP)$\rightarrow$ v is visc. sol $\rightarrow$ v is smooth sol $\rightarrow$ (DPP) $+$opt. controls

Examples: Shreve and Soner, Pham

Viscosity solution is smooth, cont'd

- the first step is hardest to prove
- the program seems circular

Question: can we just avoid the first step, proving the (DPP)?
Answer: yes, we can use (Ishii's version of) Perron's method to construct (quite easily) a viscosity solution.

Lucky case, revisited

Perron $\rightarrow$ visc. sol $\rightarrow$ smooth sol $\rightarrow$ unique+(DPP) +opt. controls

Example: Janeček, S.

Comments:

- old news for PDE
- the new approach is analytic/direct

Perron's method

General Statement: sup over sub-solutions and inf over super-solutions are solutions.

$$v^- = \sup_{w \in \mathscr{U}^-} w, v^+ = \inf_{w \in \mathscr{U}^+} w \textit{ are solutions}$$

Ishii's version of Perron (1984): sup over viscosity sub-solutions and inf over viscosity super-solutions are viscosity solutions.

$$v^- = \sup_{w \in \mathscr{U}^{-,visc}} w, v^+ = \inf_{w \in \mathscr{U}^{+,visc}} w \textit{ are viscosity solutions}$$

Question: why not inf/sup over classical super/sub-solutions?
Answer: Because one cannot prove (in general/directly) the result is a viscosity solution. The classical solutions are not enough (the set of classical solutions is not stable under max or min).
Relation to the work to Fleming-Vermes:will get back.

Back to Objective

Provide a method/tool to replace the program

existence of smooth solution $\rightarrow$ uniqueness+(DPP) +opt. controls

in case one does not expect smoothness, by

New method/tool $\rightarrow$ **construct** a visc. sol $u \rightarrow u = v$ +(DPP)

Back to the Objective cont'd

We therefore want to replace the probabilistic approach program

(DPP)$\rightarrow$ v visc. sol.+comparison $\rightarrow$ v is the unique visc sol.

by a "direct" approach, resembling the classic/analytic one,

Constructive method $\rightarrow$ a visc. sol u +comp. $\rightarrow$ $u = v$ +(DPP)

Having in mind the "lucky case"

Perron $\rightarrow$ visc. sol $\rightarrow$ smooth sol $\rightarrow$ unique+(DPP) +opt. controls

why not try a modification of Perron's method for the constructive method?

Some comments (my understanding)

Attempting to prove first the (DPP) is mostly due to historical reasons. For deterministic control problems, proving the (DPP) is very easy; uniqueness/comparison of viscosity solutions is the most important.

In the stochastic case, the (DPP) is highly non-trivial, and a comparison result is needed anyway on top of it.

Perron's method, recall

(Ishii's version) Provides viscosity solutions of the HJB

$$v^- = \sup_{w \in \mathscr{U}^{-,visc}} w, v^+ = \inf_{w \in \mathscr{U}^{+,visc}} w$$

Problem:

- w does NOT compare to the value function v UNLESS one proves v is a viscosity solutions already AND the viscosity comparison
- if we ask w to be classical semi-solutions, we cannot prove that the inf/sup are viscosity solutions

Main Idea

Perform Perron's Method over a class of semi-solutions which are

- weak enough to conclude (in general/directly) that v^-, v^+ are viscosity solutions
- strong enough to compare with the value function **without studying the properties of the value function**

We know that

classical sol $\rightarrow$ (DPP) $\rightarrow$ viscosity sol

Actually, we have

classical semi-sol $\rightarrow$ half-(DPP) $\rightarrow$ viscosity semi-sol

The idea: | half (DPP)= stochastic semi-solution |

Main property: stochastic sub and super-solutions DO compare with the value function v!

Stochastic Perron Method, quick summary

General Statement:

- supremum over stochastic sub-solutions is a viscosity (super)-solution

$$v_* = \sup_{w \in \mathscr{U}^{-,stoch}} w \leq v$$

- infimum over stochastic super-solutions is a viscosity (sub)-solution

$$v^* = \inf_{w \in \mathscr{U}^{+,stoch}} w \geq v$$

Conclusion:

$$v_* \leq v \leq v^*$$

IF we have a viscosity comparison result, then v is the unique viscosity solution!

(SP)+visc comp $\rightarrow$ (DPP)+ v is the unique visc sol of (DPE)

Some comments

- the Stochastic Perron Method plus viscosity comparison substitute for (large part of) verification (in the analytic approach)
- this method represents a "probabilistic version of the analytic approach"
- loosely speaking, stochastic sub and super-solutions amount to sub and super-martingales
- stochastic sub and super-solution have to be carefully defined (depending on the control problem) as to obtain viscosity solutions as sup/inf (and to retain the comparison build in)

Stochastic Perron Method: the Mathematics

Completed (with E. Bayraktar) for

1. Linear Case (Proceedings of AMS)
2. Dynkin Games (Proceedings of AMS)
3. Differential Control Problems (submitted)

Seems to work fine for Differential games (in progress)

Linear case

Want to compute $v(s,x) = \mathbb{E}[g(X_T^{s,x})]$, for

$$\begin{cases} dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t \\ X_s = x. \end{cases}$$

Assumption: continuous coefficients with linear growth
There exist (possibly non-unique) weak solutions of the SDE.

$$\Big((X_t^{s,x})_{s\le t\le T}, (W_t^{s,x})_{s\le t\le T}, \Omega^{s,x}, \mathscr{F}^{s,x}, \mathbb{P}^{s,x}, (\mathscr{F}_t^{s,x})_{s\le t\le T}\Big),$$

where the $W^{s,x}$ is a d-dimensional Brownian motion on the stochastic basis

$$(\Omega^{s,x}, \mathscr{F}^{s,x}, \mathbb{P}^{s,x}, (\mathscr{F}_t^{s,x})_{s\le t\le T})$$

and the filtration $(\mathscr{F}_t^{s,x})_{s\le t\le T}$ satisfies the usual conditions. We denote by $\mathscr{X}^{s,x}$ the non-empty set of such weak solutions.

Which selection of weak solutions to consider?

Just take sup/inf over all solutions.

$$v_*(s,x) := \inf_{X^{s,x} \in \mathscr{X}^{s,x}} \mathbb{E}^{s,x}[g(X_T^{s,x})]$$

and

$$v^*(s,x) := \sup_{X^{s,x} \in \mathscr{X}^{s,x}} \mathbb{E}^{s,x}[g(X_T^{s,x})].$$

The (linear) PDE associated

$$\begin{cases} -v_t - L_t v = 0 \\ v(T,x) = g(x), \end{cases} \tag{1}$$

Assumption: g is bounded (and measurable).

Stochastic sub and super-solutions

Definition

A stochastic sub-solution of (1) $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$

1. lower semicontinuous (LSC) and bounded on $[0,T] \times \mathbb{R}^d$. In addition $u(T,x) \le g(x)$ for all $x \in \mathbb{R}^d$.
2. for each $(s,x) \in [0,T] \times \mathbb{R}^d$, and each weak solution $X^{s,x} \in \mathscr{X}^{s,x}$, the process $(u(t, X_t^{s,x}))_{s \le t \le T}$ is a submartingale on $(\Omega^{s,x}, \mathbb{P}^{s,x})$ with respect to the filtration $(\mathscr{F}_t^{s,x})_{s \le t \le T}$.

Denote by $\mathscr{U}^-$ the set of all stochastic sub-solutions.

Semi-solutions cont'd

Symmetric definition for stochastic super-solutions $\mathscr{U}^+$.

Definition

A stochastic super-solution $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$

1. upper semicontinuous (USC) and bounded on $[0, T] \times \mathbb{R}^d$. In addition $u(T, x) \geq g(x)$ for all $x \in \mathbb{R}^d$.
2. for each $(s, x) \in [0, T] \times \mathbb{R}^d$, and each weak solution $X^{s,x} \in \mathscr{X}^{s,x}$, the process $(u(t, X_t^{s,x}))_{s \leq t \leq T}$ is a supermartingale on $(\Omega^{s,x}, \mathbb{P}^{s,x})$ with respect to the filtration $(\mathscr{F}_t^{s,x})_{s \leq t \leq T}$.

About the semi-solutions

- if one choses a Markov selection of weak solutions of the SDE (and the canonical filtration), super an sub solutions are the time-space super/sub-harmonic functions with respect to the Markov process X
- we use the name associated to Stroock–Varadhan. In Markov framework, sub+ super-solution is a stochastic solution in the definition of Stroock-Varadhan.

The definition of semi-solutions are strong enough to provide comparison to the expectation(s).
For each $u \in \mathscr{U}^-$ and each $w \in \mathscr{U}^+$ we have

$$u \leq v_* \leq v^* \leq w.$$

Define

$$v^- := \sup_{u \in \mathscr{U}^-} u \leq v_* \leq v^* \leq v^+ := \inf_{w \in \mathscr{U}^+} w.$$

We have (need to be careful about point-wise inf)

$$v^- \in \mathscr{U}^-, \quad v^+ \in \mathscr{U}^+.$$

Linear Stochastic Perron

Theorem

(Stochastic Perron's Method) *If g is bounded and LSC then v^- is a bounded and LSC viscosity supersolution of*

$$\begin{cases} -v_t - L_t v \geq 0, \\ v(T,x) \geq g(x). \end{cases} \tag{2}$$

If g is bounded and USC then v^+ is a bounded and USC viscosity subsolution of

$$\begin{cases} -v_t - L_t v \leq 0, \\ v(T,x) \leq g(x). \end{cases} \tag{3}$$

Comment: new method to construct viscosity solutions (recall v^- and v^+ are anyway stochastic sub and super-solutions).

Verification by viscosity comparison

Definition

Condition $CP(T, g)$ is satisfied if, whenever we have a bounded (USC) viscosity sub-solution u and a bounded LSC viscosity super-solution w we have $u \leq w$.

Theorem

Let g be bounded and continous. Assume $CP(T, g)$. Then there exists a unique bounded and continuous viscosity solution v to (1), and

$$v_* = v = v^*.$$

In addition, for each $(s, x) \in [0, T] \times \mathbb{R}^d$, and each weak solution $X^{s,x} \in \mathscr{X}^{s,x}$, the process $(v(t, X^{s,x}))_{s \leq t \leq T}$ is a martingale on $(\Omega^{s,x}, \mathbb{P}^{s,x})$ with respect to the filtration $(\mathscr{F}_t^{s,x})_{s \leq t \leq T}$.

Comments:

- v is a stochastic solution (in the Markov case)
- if comparison holds for all T and g, then the diffusion is actually Markov (but we never use that explicitly)

Idea of proof

To show that v^- is a super-solution

- touch v^- from below with a smooth test function φ
- if the viscosity super-solution property is violated, then φ is locally a smooth sub-solution
- push it to $\varphi_\varepsilon = \varphi + \varepsilon$ slightly above, to still keep it a smooth sub-solution (locally)
- Itô implies that φ_ε is also (locally wrt stopping times) a submartingale along X
- take $\max\{v^-, \varphi_\varepsilon\}$, still a stochastic-subsolution (need to "patch" sub-martingales along a sequence of stopping times)

Comments:

- why don't we need Markov property? Because we only use Itô, which does not require the diffusion to be Markov.
- the proof is very similar to Ishii's proof, but instead of applying the differential operator to the test function φ we apply Itô.

Nonlinear Problems

A very important part for nonlinear problems is choosing the best suited definition of stochastic semi-solution. While the intuition is obvious (write formally the DPP and choose the corresponding inequality as definition) the precise definition has to take into account that only Itô formula will be used, and not the Markov property.
In the end, it has to be done case by case, depending on the control problem.

Obstacle problems and Dynkin games

First example of non-linear problem.

Same diffusion framework as for the linear case. Choose a selection of weak solutions $X^{s,x}$ to save on notation.

$g : \mathbb{R}^d \to \mathbb{R}$, $l, u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ *bounded* and measurable, $l \leq u$, $l(T, \cdot) \leq g \leq u(T, \cdot)$.

Denote by $\mathscr{T}^{s,x}$ the set of stopping times τ (with respect to the filtration $(\mathscr{F}_t^{s,x})_{s \leq t \leq T}$) which satisfy $s \leq \tau \leq T$.

The first player (ρ) *pays* to the second player (τ) the amount

$$J(s, x, \tau, \rho) :=$$

$$= \mathbb{E}^{s,x}\left[\mathbb{I}_{\{\tau<\rho\}} l(\tau, X_\tau^{s,x}) + \mathbb{I}_{\{\rho\leq\tau,\rho<T\}}) u(\rho, X_\rho^{s,x}) + \mathbb{I}_{\{\tau=\rho=T\}} g(X_T^{s,x})\right].$$

Dynkin games, cont'd

Lower value of the Dynkin game

$$v_*(s,x) := \sup_{\tau \in \mathscr{T}^{s,x}} \inf_{\rho \in \mathscr{T}^{s,x}} J(s,x,\tau,\rho)$$

and the *upper value of the game*

$$v^*(s,x) := \inf_{\rho \in \mathscr{T}^{s,x}} \sup_{\tau \in \mathscr{T}^{s,x}} J(s,x,\tau,\rho).$$

$$v_* \leq v^*$$

Remark: we could appeal directly to what is known about Dynkin games to conclude $v_* \leq v^*$, but this is exactly what we wish to avoid.

DPE equation for Dynkin games

$$\begin{cases} F(t,x,v,v_t,v_x,v_{xx}) = 0, & \text{on } [0,T) \times \mathbb{R}^d, \\ u(T,\cdot) = g, \end{cases} \tag{4}$$

where

$$\begin{aligned} F(t,x,v,v_t,v_x,v_{xx}) &:= \\ \max\{v-u, \min\{-v_t - L_t v, v-l\}\} \\ = \min\{v-l, \max\{-v_t - L_t v, v-u\}\}. \end{aligned} \tag{5}$$

Super and Subsolutions

Definition

$\mathscr{U}^+$, is the set of functions $w : [0, T] \times \mathbb{R}^d \to \mathbb{R}$

1. are continuous (C) and bounded on $[0, T] \times \mathbb{R}^d$. $w \geq l$ and $w(T, \cdot) \geq g$.
2. for each $(s, x) \in [0, T] \times \mathbb{R}^d$, and any stopping time $\tau_1 \in \mathscr{T}^{s,x}$, the function w along the solution of the SDE is a super-martingale in between τ_1 and the first (after τ_1) hitting time of the upper stopping region $\mathscr{S}^+(w) := \{w \geq u\}$. More precisely, for any $\tau_1 \leq \tau_2 \in \mathscr{T}^{s,x}$, we have

$$w(\tau_1, X^{s,x}_{\tau_1}) \geq \mathbb{E}^{s,x}\left[w(\tau_2 \wedge \rho^+, X^{s,x}_{\tau_2 \wedge \rho^+}) | \mathscr{F}^{s,x}_{\tau_1}\right] - \mathbb{P}^{s,x} \ a.s.$$

   where the stopping time ρ^+ is defined as

$$\rho^+(v, s, x, \tau_1) = \inf\{t \in [\tau_1, T] : X^{s,x}_t \in \mathscr{S}^+(w)\}.$$

Question: why the starting stopping time? No Markov property.

Stochastic Perron for Obstacle Problems

Define symmetrically sub-solutions $\mathscr{U}^-$. Now define, again

$$v^- := \sup_{w \in \mathscr{U}^-} w \leq v_* \leq v^* \leq v^+ := \inf_{w \in \mathscr{U}^+} w.$$

Cannot show $v^- \in \mathscr{U}^-$ or $v^+ \in \mathscr{U}^+$, but it is not really needed. All is needed is stability with respect to max/min, not sup/inf (and this is the reason why we can assume continuity).

Theorem

- *v^- is viscosity super-solution of the (DPE)*
- *v^+ is viscosity sub-solution of the (DPE)*

Verification by comparison for obstacle problems

Theorem

- *if comparison holds, then there exists a unique and continuous viscosity solution v, equal to $v^- = v_* = v^* = v^+$*
- *the first hitting times are optimal for both players*

In the Markov case, Peskir showed (with different definitions for sub, super-solutions, which actually involve the value function) that

$$v^- = v^+$$

by showing that $v^- =$ "*value function*" $= v^+$. Peskir generalizes the characterization of value function in optimal stopping problems.

What about optimal stopping $u = \infty$?

Classic work of El Karoui, Shiryaev: in the Markov case, the value function is the least excessive function. In our notation

$$v^+ := \inf_{w \in \mathscr{U}^+} w = v.$$

Comment: the proof requires to actually show that $v \in \mathscr{U}^+$. We avoid that, showing that

$$v^- \leq v \leq v^+,$$

and then using comparison.
We provide a short cut to conclude the value function is the continuous viscosity solution of the free-boundary problem (study of continuity in Bassan and Ceci)

Back to the Original Differential Control Problem

$$v(s,x) = \sup_{\alpha} \mathbb{E}[\int_s^T R(t, X_t, \alpha_t)dt + g(X_T)],$$

subject to

$$\begin{cases} dX_t = b(t, X_t, \alpha_t)dt + \sigma(t, X_t, \alpha_t)dW_t \\ X_s = x. \end{cases}$$

Stochastic Semi-Solutions for Control Problems

Definition (Super-solutions, easier)

$\mathscr{U}^+$, is the set of functions $w : [0, T] \times \mathbb{R}^d \to \mathbb{R}$

1. are continuous (C) and satisfy some bounds, $w(T, \cdot) \geq g$.
2. for each $(s, x) \in [0, T] \times \mathbb{R}^d$, and any control α,

$$(w(t, X^{s,x;\alpha})_t)_{s \leq t \leq T}$$

   is a super-martingale.

Definition (Sub-solutions, more delicate)

$\mathscr{U}^-$, is the set of functions $w : [0, T] \times \mathbb{R}^d \to \mathbb{R}$

1. are continuous (C) and satisfy some bounds, $w(T, \cdot) \leq g$.
2. for each stopping time τ and any $\xi \in \mathscr{F}_\tau$, there exists a control α (starting at τ) such that

$$w(\tau, \xi)_{\leq} \mathbb{E}[w(\rho, X_\rho^{\tau,\xi;\alpha} | \mathscr{F}_\tau], \quad \forall \tau \leq \rho \leq T$$

Stochastic Perrron for HJB's

Define

$$v^- := \sup_{u \in \mathscr{U}^-} u \leq v \leq v^+ := \inf_{w \in \mathscr{U}^+} w.$$

Theorem

1. v^+ *viscosity sub-solution*
2. v^- *viscosity super-solution*

If we also have comparison, we are done!

Fleming and Vermes: approximate the optimal control and use a separation argument to show, under some conditions, that

$$v = \inf_{w \in \mathscr{U}^+_{classical}} w.$$

- it implies directly that $v^+ = v$.
- however, by itself, it only shows that the value function is a viscosity super-solution. We still need Part 1 from the Theorem above to get v is a viscosity solution
- one-sided argument (does not work on games)

Conclusions

- new method to construct viscosity solutions as sup/inf of stochastic sub/super-solutions
- compare directly with the value function
- if we have viscosity comparison, then the value fct is the unique continuous solution of the (DPE) and the (DPP) holds

Conjecture

Any PDE that is associated to a stochastic optimization problem can be approached by Stochastic Perron's Method.

Actually, differential games work out fine, at least when the Isaacs condition holds (work in progress).