# Stochastic Perron's Method and Elementary Strategies for Differential Games

Mihai Sîrbu, The University of Texas at Austin

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# Outline

# Objective

- take a new look at the problem of two-player zero-sum stochastic differential games
- using the Stochastic Perron's Method

# Stochastic Perron's Method

**Meta-Theorem:** The value function of a Markovian stochastic optimization problem is the unique viscosity solution of the Dynamic Programming Equation (DPE).

**Stochastic Perron:** a method to prove this statement, in different cases, **avoiding the proof of the Dynamic Programming Principle (DPP)**.
Comment: in the stochastic case, the DPP is non-trivial to prove, unlike the deterministic case.

# Stochastic Perron: main idea

- write the DPP formally
- define stochastic sub and super-solutions as functions that satisfy (roughly) half of the DPP (one side)
- with this definitions, sub-solutions lie below the value functions and super-solutions above
- consider inf of super-solutions and sup of sub-solutions (Perron), obtaining

$$\text{sup of sub-solutions} \triangleq v^- \leq V \leq v^+ \triangleq \text{inf of super-solutions}$$

- show that $v^-$ is viscosity super-solution and $v^+$ is viscosity sub-solution
- comparison then tells

$$v^- = V = v^+ = \text{unique cont. visc. sol}$$

# Some comments on Stochastic Perron's Method

- the comparison result has to be proven anyway (to have stability of numerical schemes)
- the proof that $v^-$ and $v^+$ are quite elementary, and rely only on the definitions of stochastic and viscosity semi-solutions
- the proofs are similar to those of Ishii, but instead of placing derivatives on test functions, we apply Itô (to see soon)

Additional comments:

- the method is constructive/direct like the classic "smooth solution + verification program"
- the properties of value function are not studied using its definition
- actually, this is a probabilistic version of classic dynamic programming approach (above)

# Stochastic Perron: cases when it works so far

(joint with E. Bayraktar)

- linear case (Proc. of AMS)
- optimal stopping/Dynkin games (to appear Proc. of AMS)
- control problems and HJB (submitted)

# SP: linear vs. non-linear problems

1. linear case
   - defining the problem (compute expectation of a payoff from a diffusion)
   - defining stochastic semi-solutions (sub or super-martingales along such diffusion)

   are both "canonical"
2. for the non-linear problems studied
   - stating the problem is "canonical" (optimal stopping, control problem)
   - defining stochastic sub and super-solutions is flexible, and this choice is very important. Basically, one has to choose any definition that allows direct comparison with the value function and keeps the proofs correct.

# What about games?

- there is no widely accepted notion of strategy/control, so no "canonical" formulation anyway (Cardaliaguet lecture notes)
- proving Dynamic Programming Principle does not work very well (see Fleming-Souganidis)

The idea: approach differential games using Stochastic Perron's Method. This means

- define (precisely) the game
- define stochastic semi-solutions

# Continuous time zero-sum differential games

Very selective list of works

- Isaacs: deterministic case
- Elliott-Kalton: deterministic case, use so called strategies for the stronger player, (open loop) controls for weaker player (no symmetric formulation)
- Fleming-Souganidis: use Elliott-Kalton strategies in stochastic games. Prove the value functions are viscosity solutions of DPE (Bellman-Isaacs equations). No direct proof of DPP. No symmetric formulation.
- large interesting literature on games studied using BSDE's: Hamadene and Lepeltier, El Karoui-Hamadene, Buckdahn-Li (more others)

# Literature cont'd

Most definitions of a game are not-symmetric: the two values don't compare by definition, but only after complicated analysis. This is the case if the stronger player uses Elliott-Kalton strategies and the other (open loop) controls.

Recently, symmetric formulation of games where both players use strategies based only on the knowledge of the past of the state have been considered:

- Cardaliaguet-Rainer '08 (strong formulation, path-wise feedback strategies with delay)
- Pham -Zhang '12 (path-wise feedback strategies, discretized in time and space, called "feed-back controls")

Or work fits into this last line of research (in terms of how the game is defined).

# Zero-sum differential games: inputs/rules

- a (fixed) probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a Brownian motion $W$. Strong formulation, take the filtrations generated by the increments $W$, $(\mathscr{F}_t^s)_{s \le s \le T}$.
- the coefficients of the state equation
- the sets where the two players can take action: $u \in U$, $v \in V$

The state equation generated by these inputs/rules:

$$\begin{cases} dX_t = b(t, X_t, u_t, v_t)dt + \sigma(t, X_t, u_t, v_t)dW_t, \\ X_s = x \in \mathbb{R}^d, \end{cases}$$

starting at initial time $s$ at position $x$, and which is controlled by both players. The second player ($v$) pays to the first player ($u$) the amount $g(X_T^{s,x;u,v})]$ at time $T$.

**Formal** zero-sum game

$$\sup_u \inf_v \mathbb{E}[g(X_T^{s,x;u,v})], \quad \inf_v \sup_u \mathbb{E}[g(X_T^{s,x;u,v})].$$

# Making the game precise

Need to state precisely what we mean by strategies/controls $u, v$ so that

- so that the state equation makes sense
- the sup/inf problem makes sense as well

It does not make sense to use (open loop) controls, i.e. adapted processes $u, v$.

Possible formulation: Fleming-Souganidis following Elliott-Kalton

- the weaker player chooses an (open loop) control
- the stronger player observes the weaker player's control and chooses a strategy, which is a mapping from the set of weaker player's controls to the set of stronger player controls (non-anticipative)

# Elliott-Kalton strategies

- it means that the rules of the game give an advantage to one of the players by observing the other player's actions
- the formulation of the game is non-symmetric
- the two values do not compare by definition
- the stronger player is on the "exterior" of inf/sup or sup/inf
- if the game does have a value, the (approximate) saddle points consist of strategies. The state equation does not really make sense when we plug in two such strategies.

# Feed-back strategies

(considered in Cardaliaguet-Rainer, Pham-Zhang)
Both players

- observe the state only
- do not not the other player's actions
- do not observe the noise

Therefore, if $C([s, T])$ is the path-space for the state, we can consider

$$u : (s, T] \times C([s, T]) \to U$$

and

$$v : (s, T] \times C([s, T]) \to V$$

adapted to the filtration $\mathbb{B} = (\mathscr{B}_t)_{s \leq t \leq T}$ defined by

$$\mathscr{B}_t^s \triangleq \sigma(y(u), s \leq u \leq t), \quad 0 \leq t \leq T.$$

We denote by $y$ the paths.

# The state equation with feed-back strategies

Choose $u(\cdot, X_\cdot)$ and $v(\cdot, X_\cdot)$ strategies: the state equation is just a (path-dependent) SDE

$$\begin{cases} dX_t = b(t, X_t, u(t, X_\cdot), v(t, X_\cdot)dt + \sigma(t, X_t, u(t, X_\cdot), v(t, X_\cdot)dW_t, \\ X_s = x \in \mathbb{R}^d, \end{cases}$$

- Cardaliaguet-Rainer: consider delayed strategies and obtain strong solutions
- Pham-Zhang: look at the weak formulation and restrict strategies to be finite valued (discrete-time, discrete-space)

The games become static/symmetric: player 1 chooses $u(\cdot, X_\cdot)$ to maximize and player 2 chooses $v(\cdot, X_\cdot)$ to minimize the payoff.

# Feed-back strategies heuristics

Consider the upper Isaacs equation (formally describing the upper value)

$$h_t + \inf_v \sup_u L^{u,v} h = 0, h(T, \cdot) = g.$$

Formally (if there is a solution $h$ smooth) we have $v(t,x) = \arg\max$ and $u(t,x,v) = \arg\min$ should be the actions of the two players. The stronger player does need the knowledge of the weaker player's control, but, if the weaker player is using a feed-back strategy than $u$ becomes a feed-back strategy as well. Heuristically, it is all the same for the stronger player to either

- observe the other player's control (Elliott-Kalton) or
- have knowledge of the other player's feedback-strategy (like we plan to set up the game)

# Elementary (feed-back, pure) strategies, definition

Restrict feed-back strategies to be constant in between a finite number of stopping time strategies (as in Karatzas and Sudderth) i.e. mappings

$$\tau : C([s, T]) \to [s, T]$$

satisfying

$$\{\tau \leq t\} \in \mathscr{B}_t^s \triangleq \sigma(y(u), s \leq u \leq t) \; \forall \; s \leq t \leq T$$

More precisely, an **elementary** strategy $u$ starting at $s$, for the first player, is defined by

- a finite non-decreasing sequence of stopping time strategies , i.e. $\tau_k \in \mathbb{B}^s$ for $k = 1, \dots, n$ and

$$s = \tau_0 \leq \dots \tau_k \leq \dots \leq \tau_n = T$$

- for each $k = 1 \dots n$, a constant value of the strategy $\xi_k$ in between the times $\tau_{k-1}$ and $\tau_k$, which is decided based only on the knowledge of the past state up to $\tau_{k-1}$, i.e. $\xi_k : C([s, T]) \to U$ such that $\xi_k \in \mathscr{B}^s_{\tau_{k-1}}$.

# Elementary strategies cont'd

The strategy is to hold $\xi_k$ in between $(\tau_{k-1}, \tau_k]$, i.e.
$u : (s, T] \times C([s, T]) \to U$ is defined by

$$u(t, y(\cdot)) \triangleq \sum_{k=1}^{n} \xi_k(y(\cdot)) 1_{\{\tau_{k-1}(y(\cdot)) < t \leq \tau_k(y(\cdot))\}}.$$

A simple strategy $v$ for the second player is defined in an identical way, but takes values in $V$. We denote by $\mathscr{U}(s, s)$ and $\mathscr{V}(s, s)$ the collections of all possible strategies for the first, and the second player, given the initial deterministic time $s$.

# The state equation with elementary strategies

We assume that the stochastic system satisfies a uniform Lipschitz condition, i.e.

$$|b(t,x,u,v)-b(t,y,u,v)|+|\sigma(t,x,u,v)-\sigma(t,y,u,v)|\leq L|x-y|$$

$\forall\ x,y\in\mathbb{R}^d, t\in[0,T]\ u\in U, v\in V$. Therefore, as long as actions of the players do not change, the state equation has a strong solution. This idea can be iterated, so that, if $u\in\mathscr{U}(s,s)$ and $v\in\mathscr{V}(s,s)$ the state equation

$$\begin{cases} dX_t = b(t,X_t,u(t,X_\cdot),v(t,X_\cdot)dt+\sigma(t,X_t,u(t,X_\cdot),v(t,X_\cdot)dW_t,\\ X_s = x\in\mathbb{R}^d,\end{cases}$$

has a unique strong solution $(X_t^{s,x;u,v})_{s\leq t\leq T}$.

# Why strong solutions for the state eq.?

We believe it is important, that, once the two players have decided on what strategy to follow, to have a one-to one map between the noise and the state.

# Our precise definition of the game

fixed $0 \leq s \leq t \leq T$ and $x \in \mathbb{R}^d$ (deterministic), we define the lower and the upper value of the game

$$V^-(s,x) \triangleq \sup_{u \in \mathscr{U}(s,s)} \inf_{v \in \mathscr{V}(s,s)} \mathbb{E}[g(X_T^{s,x:u,v})] \leq$$

$$\inf_{v \in \mathscr{V}(s,s)} \sup_{u \in \mathscr{U}(s,s)} \mathbb{E}[g(X_T^{s,x;u,v})] \triangleq V^+(s,x).$$

Since both our players play strategies (symmetrically), the lower and the upper value compare *by definition*.

**Comments:** if the game does not have a value over pure strategies one should simply proceed to considering mixed strategies (in progress). We analyze the upper and lower values still for mathematical reasons.

# Elementary Strategies starting later

Fix $s$ and let $\tau \in \mathbb{B}^s$ be a stopping time strategy. An elementary strategy, denoted by $u \in \mathscr{U}(s, \tau)$, for the first player, starting at $\tau$, is defined by

- (again) a finite non-decreasing sequence of stopping time strategies , i.e. $\tau_k \in \mathbb{B}^s$, $k = 1, \dots n$ for some finite $n$, and with $\tau = \tau_0 \leq \dots \tau_k \leq \dots \leq \tau_n = T$.
- for each $k = 1 \dots n$, a constant strategy $\xi_k$ in between the times $\tau_{k-1}$ and $\tau_k$, which is decided based only on the knowledge of the past state up $\tau_{k-1}$, i.e. $\xi_k : C([s, T]) \to U$ such that $\xi_k \in \mathscr{B}^s_{\tau_{k-1}}$.

The strategy is, again, to hold $\xi_k$ in between $(\tau_{k-1}, \tau_k]$, i.e..

$$u : \{(t, y) | \tau(y) < t \leq T, y \in C([s, T])\} \to U$$

with

$$u(t, y(\cdot)) \triangleq \sum_{k=1}^{n} \xi_k(y(\cdot)) 1_{\{\tau_{k-1}(y(\cdot)) < t \leq \tau_k(y(\cdot))\}}.$$

We define similarly $\mathscr{V}(s, \tau)$ for the second player.

# Concatenating strategies

Fix $s$ and let $\tau \in \mathbb{B}^s$ be a stopping time strategy and $\tilde{u} \in \mathscr{U}(s, \tau)$. Then, for each $u \in \mathscr{U}(s, s)$, the mapping $u \otimes_\tau \tilde{u} : (s, T] \times C([s, T]) \to U$ defined by

$$\big(u \otimes_\tau \tilde{u}\big)(t, y(\cdot)) \triangleq u(t, y(\cdot)) 1_{\{s < t \leq \tau(y(\cdot))\}} + \tilde{u}(t, y(\cdot)) 1_{\{\tau(y(\cdot)) < t \leq T\}}$$

is a simple strategy starting at $s$, i.e. $u \otimes_\tau \tilde{u} \in \mathscr{U}(s, s)$. A similar statement holds for the second player.

# Stochastic super-solutions of upper Isaacs equation

A function $w : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is called stochastic super-solution of the upper Isaacs equation if

1. it is bounded, continuous and $w(T, \cdot) \geq g(\cdot)$,
2. for each $s$ and for each stopping time strategy $\tau : C([s, T]) \to [s, T]$, $\tau \in \mathbb{B}^s$ there exists an elementary strategy $\tilde{v} \in \mathscr{V}(s, \tau)$ such that, for any $u \in \mathscr{U}(s, s)$ any $v \in \mathscr{V}(s, s)$, any $x \in \mathbb{R}^d$ and each stopping time strategy $\tau \leq \rho \leq T$, $\rho \in \mathbb{B}^s$, with the simplifying notations

   $$X \triangleq X^{s,x,u,v \otimes_\tau \tilde{v}}, \tau' \triangleq \tau(X), \rho' \triangleq \rho(X),$$

   we have

   $$w(\tau', X_{\tau'}) \geq \mathbb{E}[w(\rho', X_{\rho'}) | \mathscr{F}^s_{\tau'}] \;\; \mathbb{P} - a.s.$$

# Stoch. super-sol. of upper Isaacs cont'd

Choosing $\tau = s$, we can see that, if $w$ is a stochastic super-solution, there exists $\tilde{v} \in \mathscr{V}(s,s)$ such that

$$w(s,x) \geq \mathbb{E}\left[ w(\rho(X_\cdot^{s,x,u,\tilde{v}}), X_{\rho(X_\cdot^{s,x,u,\tilde{v}})}^{s,x,u,\tilde{v}} | \mathscr{F}_s^s \right], \ \mathbb{P} - a.s$$

for all $u \in \mathscr{U}(s,s)$ and $\rho \in \mathbb{B}^s$. After taking the expectation, it is now obvious that, if $w$ is a stochastic super-solution, then we have the **half DPP** for $w$, i.e.

$$w(s,x) \geq \inf_{v \in \mathscr{V}(s,s)} \sup_{u \in \mathscr{U}(s,s)} \mathbb{E}\left[ w(\rho(X_\cdot^{s,x,u,v}), X_{\rho(X_\cdot^{s,x,u,v})}^{s,x,u,v}) \right], \quad \forall \rho \in \mathbb{B}^s.$$

Since $w(T,\cdot) \geq g(\cdot)$, we obtain $w(s,x) \geq V^+(s,x)$.

# Stochastic sub-solutions of upper Isaacs equation

A function $w : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is called stochastic sub-solution of the upper Isaacs equation if

1. it is bounded, continuous and $w(T, \cdot) \leq g(\cdot)$,
2. for each $s$ and for each stopping time strategy $\tau : C([s, T] \to [s, T]$, $\tau \in \mathbb{B}^s$ and each strategy $v \in \mathscr{V}(s, s)$ there exists an elementary strategy $\tilde{u} \in \mathscr{U}(s, \tau)$ (depending on $v$ and $\tau$) such that, for any $u \in \mathscr{U}(s, s)$ and any $x$ as well as each stopping time strategy $\tau \leq \rho \leq T$, $\rho \in \mathbb{B}^s$, with the simplifying notation

$$X \triangleq X^{s,x,u \otimes_\tau \tilde{u}, v}, \tau' \triangleq \tau(X), \rho' \triangleq \rho(X),$$

we have

$$w(\tau', X_{\tau'}) \leq \mathbb{E}[w(\rho', X_{\rho'}) | \mathscr{F}^s_{\tau'}] \;\; \mathbb{P} - a.s.$$

# Stoch. sub-sol. of upper Isaacs cont'd

Let $w$ a stochastic sub-solution of upper Isaacs. Fix $v \in \mathscr{V}(s,s)$ and $\tau = s$. There exists $\tilde{u} \in \mathscr{U}(s,s)$ (depending on $v$) such that, for each $\rho \in \mathbb{B}^s$ we have

$$w(s,x) \le \mathbb{E}\left[ w(\rho(X_\cdot^{s,x,\tilde{u},v}), X_{\rho(X_\cdot^{s,x,\tilde{u},v})}^{s,x,\tilde{u},v} | \mathscr{F}_s^s \right], \ \mathbb{P} - a.s$$

After taking the expectation, it is now obvious that, if $w$ is a stochastic super-solution, then we have the **other half DPP** for $w$, i.e.

$$w(s,x) \le \inf_{v \in \mathscr{V}(s,s)} \sup_{u \in \mathscr{U}(s,s)} \mathbb{E}\left[ w(\rho(X_\cdot^{s,x,u,v}), X_{\rho(X_\cdot^{s,x,u,v})}^{s,x,u,v}) \right], \quad \forall \rho \in \mathbb{B}^s.$$

Since $w(T,\cdot) \le g(\cdot)$, we obtain easily that $w(s,x) \le V^+(s,x)$.

# Stochastic semi-solutions of lower Isaacs equation

are defined symmetrically, in obvious fashion. We denote by

1. $\mathscr{U}^+$ the set of stochastic super-solutions of upper Isaacs
2. $\mathscr{U}^-$ the set of stochastic sub-solutions of upper Isaacs
3. $\mathscr{L}^+$ the set of stochastic super-solutions of lower Isaacs (not defined explicitly)
4. $\mathscr{L}^-$ the set of stochastic sub-solutions of lower Isaacs.

Since $g$ is (assumed) bounded, all the sets are obviously non-empty

# Stochastic Perron for games

Take sup of sub-solutions andinf of super-solutions (by which we mean stochastic semi-solutions):

$$v^- \triangleq \sup_{w \in \mathscr{U}^-} \leq V^+ \leq \inf_{w \in \mathscr{U}^+} w \triangleq v^+$$

and

$$w^- \triangleq \sup_{w \in \mathscr{L}^-} \leq V^- \leq \inf_{w \in \mathscr{L}^+} w \triangleq w^+.$$

We can say, without need for any more proof, that $w^- \leq V^- \leq V^+ \leq v^+$.

# Standing assumptions

- $g$ is continuous and bounded,
- $b$ and $\sigma$ are continuous on their whole corresponding domains and uniformly Lipschitz in $x$.
- $U, V$ are compact

# Main Results 1

The function $v^+$ is a bounded upper semi-continuous (USC) viscosity sub-solution of the upper Isaacs equation

$$\begin{cases} h_t + \inf_v \sup_u L^{u,v} h = 0 \\ h(T, \cdot) = g \end{cases}$$

and satisfies the Half DPP

$$v^+(s,x) \geq \inf_{v \in \mathscr{V}(s,s)} \sup_{u \in \mathscr{U}(s,s)} \mathbb{E}\left[ v^+(\rho(X_\cdot^{s,x,u,v}), X_{\rho(X_\cdot^{s,x,u,v})}^{s,x,u,v} \right], \quad \forall \rho \in \mathbb{B}^s.$$

The function $v^-$ is a bounded lower semi-continuous (LSC) viscosity super-solution of the upper Isaacs equation and satisfies the half DPP

$$v^-(s,x) \leq \inf_{v \in \mathscr{V}(s,s)} \sup_{u \in \mathscr{U}(s,s)} \mathbb{E}\left[ v^-(\rho(X_\cdot^{s,x,u,v}), X_{\rho(X_\cdot^{s,x,u,v})}^{s,x,u,v} \right], \quad \forall \rho \in \mathbb{B}^s.$$

# Main results 1'

The function $w^+$ is a bounded upper semi-continuous (USC) viscosity sub-solution of the lower Isaacs equation, and the function $w^-$ is a bounded lower semi-continuous (LSC) viscosity super-solution of the lower Isaacs equation

$$\begin{cases} h_t + \sup_u \inf_v L^{u,v} h = 0 \\ h(T, \cdot) = g \end{cases}$$

(and they both satisfy the corresponding half of the DPP for the lower equation)

# Main results 2

The upper and the lower Isaacs equation satisfy a comparison principle between (bounded, semi-continuous) viscosity semi-solutions. Therefore

- $v^- = V^+ = v^+$ is the unique continuous viscosity solution of the upper Isaacs equation. In addition, the upper value $V^+$ satisfies the (DPP)

$$V^+(s,x) = \inf_{v \in \mathscr{V}(s,s)} \sup_{u \in \mathscr{U}(s,s)} \mathbb{E}\left[ V^+(\rho(X_\cdot^{s,x,u,v}), X^{s,x,u,v}_{\rho(X_\cdot^{s,x,u,v})}\right], \forall \rho \in \mathbb{B}^s$$

- $w^- = V^- = w^+$ is the unique continuous viscosity solution of the lower Isaacs equation. In addition, the lower value $V^-$ satisfies the (DPP)

$$V^-(s,x) = \sup_{u \in \mathscr{U}(s,s)} \inf_{v \in \mathscr{V}(s,s)} \mathbb{E}\left[ V^-(\rho(X_\cdot^{s,x,u,v}), X^{s,x,u,v}_{\rho(X_\cdot^{s,x,u,v})}\right], \forall \rho \in \mathbb{B}^s$$

# Main results 3

If, in addition, the Isaacs condition hold

$$\sup_u \inf_v L^{u,v} h = \inf_v \sup_u L^{u,v} h$$

then, the game has a value. There exist $\varepsilon$-saddle points within the class of elementary (pure) strategies.

**Comments:** Actually, in this case, one has to do only half of the work (and actually the easier half).

# Conclusions

- Stochastic Perron's Method works well for games. It amounts to a verification result for non-smotth viscosity solutions using viscosity comparison. It is a direct/constructive method.
- games are quite different from control problems, and the Stochastic Perron's method differs in significant ways
- it is important how we define both the game and the stochastic semi-solutions
- we define the game symmetrically, over pure elementary feed-back strategies (state equation has strong solutions)
- we have flexibility in defining stochastic sub and super-solutions (to make the proofs work, but keep comparison with the value function obvious)
- once the game is set up and stochastic semi-solutions are defined, the proofs are quite elementary, and amount to applying Itô to test functions in the definition of viscosity solutions

# Conclusions cont'd

- if the Isaacs condition don't hold, the game over pure strategies is a mathematical exercise (which we go over)
- the (DPP) for games holds over stopping time strategies (Karatzas and Sudderth), not stopping times on the physical probability space

# Work in progress

Consider mixed elementary feed-back strategies and obtain a game that has a value, which is the solution of the unified/mixed Isaacs equation.