Asymptotic analysis of utility-based prices and hedging strategies for utilities defined on the whole real line

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April 6th 2009

Outline

Optimal investment and utility-based pricing hedging

Asymptotic expansions

Summary

The financial model

1. there are $d+1$ traded (liquid) assets:
   - money market account B. We assume the interest rate $r=0$: $B=1$
   - d stocks: $S=(S^1,\dots,S^d)$ (semimartingale on the stochastic basis $(\Omega,\mathscr{F},(\mathscr{F}_t)_{0\le t\le T},\mathbb{P})$
2. N non-traded or illiquid European contingent claims with:
   - maturity T
   - payoff $f=(f_i)_{1\le i\le N}$

Think $N=1$ for simplicity of notation

The economic agent

1. position (x, q) at time 0:
   - initial capital x, invested in money market and stocks
   - q units of contingent claims f
2. time horizon T
3. preferences over terminal wealth described by a utility function U

Trading strategies and optimal investment

invests initial (liquid) wealth x holding H_t stocks at any time t (H is predictable and S-integrable)

- (liquid) wealth process

$$X_t = x + \int_0^t H_u dS_u$$

  $\mathscr{X}(x)$ is set of wealth processes with initial capital x (subject to some restrictions depending of the kind of utility)

- Total wealth at maturity: $X_T + qf$

Optimal investment with random endowment:

$$u(x,q) = \sup_{X \in \mathscr{X}(x)} \mathbb{E}\left[U(X_T + qf)\right]$$

Denote by $X(x,q)$ the optimal trading strategy above
Remark: when $q = 0$ we have the special case of "pure investment"

$$u(x) := u(x,0), \; X(x) := X(x,q)$$

Utility-based pricing and hedging

Investor with initial position (x, q)

- prices depend on preferences and position (x, q)
- hedging = trading strategy that offsets the risk coming from the contingent claims
- measure risk/return using utility functions
- hedging (of the q contingent claims) is embedded in the problem of optimal investment with contingent claims: Hodges & Neuberger, Davis, Duffie et al., Henderson, Hobson, etc

Definition of utility-based prices

Definition 1: the (vector) $p = p(x, q)$ is called utility based price for position (x, q) if

$$u(x, q) \geq u(\tilde{x}, \tilde{q}),$$

whenever $x + qp = \tilde{x} + \tilde{q}p$
Remark: pricing by marginal rate of substitution
Definition 2: the buyer's reservation price for q claims for an agent having x initial liquid wealth and no claims is defined by

$$u(x) = u(x - b(x, q), q).$$

Definition of hedging strategies

Definition 3: the number $c(x,q)$ is called certainty equivalent value of the position (x,q) if

$$u(c(x,q)) = u(x,q)$$

Definition 4: the utility-based hedging strategy for the q contingent claims is defined by

$$G(x,q) = X(c(x,q)) - X(x,q)$$

Remarks:

- split (by definition) the optimal investment strategy into "pure investment" and "hedging"

$$X(x,q) = X(c(x,q)) - G(x,q)$$

Approximation of prices and hedging strategies

expansion around $q = 0$ (where we can do computations)

- first order expansion of $p(x, q) \approx p(x, 0) + D(x)q$ for small q
- first order expansion for $G(x, q)$
- second order expansions for $b(x, q)$ and $c(x, q)$ for small q

Two kinds of questions:

- Quantitative: compute the expansions
- Qualitative:
  - when is $D(x)$ symmetric?
  - relate pricing to hedging
  - relate pricing/hedging to quadratic hedging

Answers to previous questions for $U : (0, \infty) \to R$

- Henderson, Henderson and Hobson: (quantitative) compute second order expansion of reservation prices for $U(x) = \frac{X^{1-p}}{1-p}$, $p > 0$ and basis risk model
- Kallsen: (quantitative and qualitative) first order expansion of utility based-prices for general utility but in the framework of local utility maximization
- Kramkov and S.: (quantitative and qualitative) general utility and general semimartingale model, characterize the qualitative behavior in terms of existence risk-tolerance wealth processes

Objective

- Answer the same questions for a utility function

$$U : (-\infty, \infty) \to R$$

Technical difference:

- $U : (0, \infty) \to R$: easier to define admissible strategies, harder dual problem
- $U(-\infty, \infty) \to R$ harder to define admissible strategies, easier dual problem

Previous work for $U : (-\infty, \infty) \to R$

For exponential utility

$$U(x) = -e^{-\gamma x}, \gamma > 0$$

compute expansion of reservation prices and hedging strategies

- Henderson: basis risk model
- Mania and Schwezer, Becherer, Kallsen and Rheinländer, Anthropelos and Zitkovic: more general model (but still has some restrictions), relate to quadratic hedging

Results: mathematical assumptions

Assumptions:

- the stock price process S is locally bounded (or sigma bounded)
- the claim f is bounded (can be relaxed)
- the absolute risk aversion of the utility function is bounded above and below

$$0 < c_1 \leq -\frac{U''(x)}{U'(x)} \leq c_2 < \infty.$$

If V is the conjugate of U

$$V(y) = \max_{x \in R} \left[U(x) - xy \right], \; y > 0$$

then

$$\mathbb{E}[V(y\frac{d\mathbb{Q}}{d\mathbb{P}})] < \infty \iff H(\mathbb{Q}/\mathbb{P}) < \infty$$

More assumptions

Denote

- $\mathscr{M}_a$ the set of absolutely continuous martingale measures,
- $\mathscr{M}_e$ the equivalent martingale measures
- $\mathscr{P}_f$ the measures $\mathbb{Q}$ with finite entropy

$$H(\mathbb{Q}|\mathbb{P}) < \infty$$

Assumption:

$$\mathscr{M}_e \cap \mathscr{P}_f \neq \emptyset$$

Back to optimal investment

Use the framework of Owen-Zitkovic, Schachermayer, six author paper to define admissible strategies as
$\mathscr{X}(x) =$ the class of stochastic integrals $X = x + \int HdS$ such that X is a supermartingale under any absolutely continuous measure $\mathbb{Q}$ with finite entropy

$$\mathbb{Q} \in \mathscr{M}_a \cap \mathscr{P}_f$$

- We have a class of admissible strategies which is independent on the utility function, as long as utility satisfies the bounds on the risk aversion
- the optimal investment with random endowment is well posed for any (x, q)
- the indirect utility $u(x)$ is two-times differentiable and

$$0 < c_1 \leq -\frac{u''(x)}{u'(x)} \leq c_2 < \infty.$$

Asymptotic pricing and hedging: the quantitative question

Theorem 1:

- Under previous assumptions, all expansions can be computed, in terms of the second order expansion of the value function $u(x, q)$
- the problem amounts to solving the quadratic optimization problem

$$\min_{X=\int HdS, H\in\mathscr{H}^2(\mathbb{Q}(y))} \mathbb{E}_{\mathbb{Q}(y)}[\frac{-U''(X_T(x))}{U'(X_T(x))}(X+f)^2]$$

where $\mathbb{Q}(y)$ is the dual measure

$$\mathbb{Q}(y) \in \mathscr{M}_e \cap \mathscr{P}_f$$

(follows from Schachermayer, Owen and Zitkovic)

Asymptotic pricing and hedging: the qualitative question(s)

All questions have positive answer if (and only if) the risk-tolerance wealth process exists

Definition 5 For fixed $x \in R$, a wealth process $R(x)$ is called **risk-tolerance wealth process** if

$$R_T(x) = -\frac{U'(X_T(x))}{U''(X_T(x))} > 0$$

Properties of $R(x)$

(in case it exists)

- it is bounded above and below; recall that

$$0 < c_1 \leq -\frac{U'(X_T(x))}{U''(X_T(x))} \leq c_2 < \infty$$

- $R_0(x) = -\frac{u'(x)}{u''(x)}$
- it is the derivative of the optimal strategy (when there are no claims):

$$\frac{R(x)}{R_0(x)} = \lim_{\Delta x \to 0} \frac{X(x + \Delta x) - X(x)}{\Delta x}$$

Existence of $R(x)$

Theorem 2 For a **fixed financial model and utility function** the following assertions are equivalent:

- the risk-tolerance wealth process $R(x)$ exists for all $x \in R$
- the dual measure $\mathbb{Q}(y)$ does not depend on $y = u'(x)$

Theorem 3 For a **fixed utility function**, the following are equivalent:

- the risk-tolerance wealth process is well defined for any financial model
- U is an exponential utility

Theorem 4 For a **fixed financial model**, the following are equivalent

- the risk-tolerance wealth process is well defined for any utility function $U : (-\infty, \infty) \to R$
- the set of martingale measures $\mathscr{M}$ admits a largest element $\hat{\mathbb{Q}}$ with respect to second order stochastic dominance

Approximation of prices and hedging strategies with risk-tolerance wealth process

Denote

$$p(x) = p(x, 0) = \mathbb{E}_{\mathbb{Q}(y)}[f]$$

The quantity $p(x)$ is the marginal prices for zero demand (Davis).
Remark: the inputs needed to compute $p(x)$ are obtained solving the "pure investment" problem only:

$$u(x) = \sup_{X \in \mathscr{X}(x)} \mathbb{E}\left[U(X_T)\right]$$

The marginal price (at $q = 0$) can be defined as a process

$$P_t(x) = \mathbb{E}_{\mathbb{Q}(y)}[f|\mathscr{F}_t], \quad 0 \leq t \leq T$$

Kunita-Watanabe decomposition of the price process

Assume that $R(x)$ exists, and use it as numéraire:

- traded assets $S^{R(x)} = \left(\frac{R_0(x)}{R(x)}, \frac{R_0(x)S}{R(x)}\right)$
- price process $\widetilde{P}(x) = \frac{R_0(x)P(x)}{R(x)}$

Adjust the measure $\mathbb{Q}(y)$ to account for the new numéraire

$$\frac{d\mathbb{Q}^{R(x)}}{d\mathbb{Q}(y)} = \frac{R_T(x)}{R_0(x)}$$

Decomposition:

$$\widetilde{P}(x) = \widetilde{M} + \widetilde{N},$$

where $\widetilde{M} = p(x) + \int K dS^{R(x)}$, and $\widetilde{N}$ is orthogonal to $S^{R(x)}$

Theorem 5 If there is a risk-tolerance wealth process, then:

- $p(x,q) \approx p(x,0) + q\frac{u''(x)}{u'(x)}\mathbb{E}_{\mathbb{Q}(y)}[\tilde{N}^2]$
- $\tilde{G}(x,q) \approx q\tilde{M}$, where $\tilde{G}(x,q)$ is the hedging strategy measured in units of risk-tolerance

Examples

1. If $U(x) = -e^{-\gamma x}$ then the risk-tolerance wealth process exists and it is constant

$$R_t(x) = \frac{1}{\gamma}, \quad 0 \leq t \leq T.$$

   - everything reduces to quadratic hedging under original numéraire and minimal entropy measure.
   - recover the results of Mania and Schwezer, Becherer, Kallsen and Rheinländer, Anthropelos and Zitkovic

2. "generalized basis risk model" : $(S, \mathscr{F}^S)$ is complete, general utility function $U : (-\infty, \infty) \to R$

Extensions

- can relax assumptions of the claims f
- can consider initial random endowment instead

$$x \to g,$$

(as in Anthropelos and Zitkovic)
However, solving the problem for $x = g$ is as hard as solving the problem for (x, q).

Overview

1. Solve the problem of “pure investment”

$$u(x) = \sup_{X \in \mathscr{X}(x)} \mathbb{E}\left[U(X_T)\right]$$

   locally around a fixed $x \in R$. Obtain $R(x)$ and pricing measure $\mathbb{Q}(y)$ from here.
2. use $R(x)$ and $\mathbb{Q}(y)$ to compute the linear approximation of marginal prices and hedging strategies for *all* contingent claims f

Remarks:

- the investment strategy in the presence of claims

$$u(x, q) = \sup_{X \in \mathscr{X}(x)} \mathbb{E}\left[U(X_T + qf)\right]$$

  (in the first order) is split into “pure investment” and hedging
- the link between the two operations is provided by $R(x)$ and $\mathbb{Q}(y)$

Summary

- similar results to the case $U : (0, \infty) \to R$ can be proved for $U : (-\infty, \infty) \to R$ under appropriate technical conditions
- pricing and hedging in incomplete markets are parts of investment strategy
- the risk-tolerance wealth process is the natural numéraire for asymptotic pricing and hedging. Utility-based hedging reduces to mean-variance hedging under the new numéraire.
- exponential utility is very peculiar since the risk-tolerance wealth processes are constant