Shadow prices and well-posedness in the problem of optimal investment and consumption with transaction costs

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based on joint work with

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Outline

Objective

- revisit the classic problem of Davis and Norman using the so called "shadow price" approach
- treat all possible values of parameters (and therefore, study the well posedness of the problem)

The (Davis and Norman) model

One risky asset (stock)

$$dS_t = S_t(\mu\, dt + \sigma\, dB_t),\ \ t \in [0, \infty) \text{ with } S_0 > 0.$$

Transaction costs: $\underline{\lambda} \in (0, 1)$ and $\overline{\lambda} > 0$:

- one gets only $\underline{S}_t = (1 - \underline{\lambda})S_t$ for one share of the stock
- pays $\overline{S}_t = (1 + \overline{\lambda})S_t$ for it

Bond pays zero interest rate: $r = 0$.

Investment/consumption strategies

(Kallsen-Muhle Karbe notation)

- φ_t^0: number of bonds held at time t (after the transaction, if any, took place)
- φ_t: number of stocks held at time t (same as above)
- c_t consumption rate

The processes φ and φ^0 are right-continuous and of finite variation and c is nonnegative and locally integrable.
Distinguish between the initial values $(\varphi_{0-}^0, \varphi_{0-})$ and the values (φ_0^0, φ_0) (after which the processes are right-continuous). Initial position

$$(\varphi_{0-}^0, \varphi_{0-}) = (\eta_B, \eta_S).$$

Remark: the initial jump appears even if there is no transaction cost.

Investment/consumption strategies cont'd

Self-financing strategy (φ^0, φ, c):

$$\varphi_t^0 = \varphi_{0-}^0 - \int_0^t \overline{S}_u d\varphi_u^{\uparrow} + \int_0^t \underline{S}_u d\varphi_u^{\downarrow} - \int_0^t c_u \, du, \tag{1}$$

where $\varphi = \varphi_{0-} + \varphi^{\uparrow} - \varphi^{\downarrow}$ is the pathwise minimal (Hahn-Jordan) decomposition of φ.
Remark: there is a *possible jumps at time zero*, as we assume that $\varphi_{0-}^{\uparrow} = \varphi_{0-}^{\downarrow} = 0$.
Admissible strategy: can always be liquidated

$$\varphi_t^0 + \varphi_t^+ \underline{S} - \varphi_t^- \overline{S}_t \geq 0, \ \forall t \geq 0- \, .$$

Optimal investment/consumption

For $p \in (-\infty, 1)$, we consider the utility function $U : [0, \infty) \to [-\infty, \infty)$ of the power (CRRA) type

$$U(c) = \begin{cases} \frac{1}{p} c^p, & c \neq 0, p \neq 0 \\ \log(c), & c \neq 0, p = 0, \end{cases} \quad \text{and} \quad U(0) = \begin{cases} 0, & p > 0, \\ -\infty, & p \leq 0 \end{cases}$$

Optimization problem:

$$u = \sup_{(\varphi^0, \varphi, c)} \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(c_t)\, dt\right],$$

where $\delta > 0$ stands for the (constant) impatience rate.

Literature

- Constantinides and Magill (76): heuristic solution
- Davis and Norman (90): analytic solution
- Shreve and Soner (94): removed some technical conditions, making use of viscosity theory. Key assumption: well posedness

$$u < \infty$$

Conclusion: there exists a wedge around the Merton proportion such that

- jump to the boundary of the wedge
- do not trade inside the wedge

Shadow prices

Introduced by

- Jouini and Kallal (95)
- Lamberton, Pham and Schweizer (98)

1. Consistent price process: $\underline{S}_t \leq \tilde{S}_t \leq \overline{S}_t$, for all $t \geq 0$. Investor with wealth $\eta_B + \eta_S \tilde{S}_0$ can certainly do better trading in $\tilde{S}$ without transaction costs, than trading in S with transaction costs.
2. If one can find a consistent price process $\tilde{S}$ such that the corresponding optimal investment strategy (without transaction costs) trades as
   - buy stocks only when $\tilde{S} = \overline{S}$
   - sell stocks only when $\tilde{S} = \underline{S}$

   then such a trading strategy is actually optimal in the problem with transaction costs.
   The price process $\tilde{S}$ is called shadow price.

Can this idea be used for the Davis and Norman problem?

Yes, in some cases

- Kallsen and Muhle Karbe, for $p = 0$. For fixed $\tilde{S}$ use the explicit solution of the log investor without transaction costs to find a shadow price
- Herczegh and Prokaj (parallel to our work): use the solution in Shreve and Soner (the primal value function) to construct a shadow price for $p \neq 1$. The shadow price is constructed in terms of the gradient of the primal value function.

Both interpret the problem as a one dimensional free-boundary problem in terms of the variable

$$\log(\frac{\pi}{1-\pi}).$$

Both require restriction on parameters: NT region is in the first quadrant (no singularity in our coming approach).

Our shadow price approach

Exploit the fact that the shadow price is a worst case scenario among consistent prices $\overline{S} \leq \tilde{S} \leq \underline{S}$.
Parametrize consistent processes $\tilde{S}$ by (θ, Σ) such that

$$d\tilde{S}_t = \tilde{S}_t(\sigma + \Sigma_t)\,(dB_t + \theta_t\,dt),\ \ \tilde{S}_0 = \tilde{s}_0.$$

Pass to the logarithmic scale $Y_t = \log(\tilde{S}_t / S_t)$, whose dynamics is given by

$$dY_t = \alpha_0(\theta_t, \Sigma_t)\,dt + \Sigma_t\,dB_t,$$

$$Y_0 = y = \log(\frac{\tilde{S}_0}{S_0}).$$

on the natural domain $Y_t \in [\underline{y}, \overline{y}]$. Here,

$$\underline{y} = \log(1 - \underline{\lambda}), \overline{y} = \log(1 + \overline{\lambda})$$

and $\alpha_0(\Sigma, \theta) = \theta\sigma - \mu - \Sigma\Big(\frac{1}{2}\Sigma + \sigma - \theta\Big)$.

Shadow prices as a game

Recall that, η_B, η_S are fixed. Fix $y \in [\overline{y}, \underline{y}]$ and θ, Σ admissible.

1. Solve the optimal investment problem for an investor with initial wealth

$$w = \eta_B + \eta_S S_0 e^y$$

i.e. maximize over π, c (proportion of investment in $\tilde{S}$ and rate of consumption) the expected discounted utility

$$\mathbb{E}[\int_0^\infty e^{-\delta t} U(c_t)].$$

2. Minimize first over (θ, Σ)
3. Minimize over $y \in [\overline{y}, \underline{y}]$.

Summarize: up to the last minimization over y, we have a problem

$$v(w, y) := \inf_{\theta, \Sigma} \sup_{\pi, c} \mathbb{E}[\int_0^\infty e^{-\delta t} U(c_t)].$$

Game cont'd

We can write the Isaacs equation for the game with a two dimensional state (W, Y). The problem actually scales as

$$v(w, y) = \frac{w^p}{p} h(y)^{1-p},$$

and the Isaacs equation reduces to a one-dimensional equation for $h(y)$, $y \in [\overline{y}, \underline{y}]$.
Actually, from the duality theory for *complete* markets, we can easily see that

$$h(y) = \inf_{(\Sigma,\theta) \in \mathscr{P}(y)} \mathbb{E}\left[\int_0^\infty e^{-\delta t} V\left(e^{\delta t} \mathscr{E}(-\theta \cdot B)_t\right) dt\right]$$

The reduced Isaacs equation for $h(y)$ is actually an HJB (this is what we use in the paper).
Remark: the sup and inf separate in the game.

The one-dimensional HJB

$$\inf_{\Sigma,\theta}\Big(\tfrac{1}{2}\Sigma^2 h''(y)+\alpha_q(\Sigma,\theta)h'(y)-\beta(\theta)h(y)+\gamma(\theta)\Big)=0,\ \ y\in(\underline{y},\overline{y})$$

$$w''(\underline{y})=w''(\overline{y})=+\infty.$$

$$q=\tfrac{p}{1-p},\ \ \alpha_q(\Sigma,\theta)=\theta\sigma-\mu-\Sigma\Big(\tfrac{1}{2}\Sigma+\sigma-\theta(1+q)\Big),$$
$$\beta(\theta)=(1+q)\Big(\delta-\tfrac{1}{2}q\theta^2\Big),\ \text{and}\ \gamma(\theta)=\begin{cases}\tfrac{1}{2}\theta^2, & p=0,\\ \operatorname{sgn}(p), & p\neq 0.\end{cases} \tag{2}$$

The boundary conditions ensure the state constraint $Y\in[\underline{y},\overline{y}]$.

Change of variable/order reduction

The HJB has the form

$$H(h'', h', h) = 0, \quad y \in [\overline{y}, \underline{y}].$$

Expect h' to be

- increasing
- negative

Change the variable to $x = -h'(y)$. The (fixed) boundaries $\underline{y}, \overline{y}$ change to

$$\underline{x} = -h'(\overline{y}), \overline{x} = -h'(\underline{y}).$$

Rewrite the HJB by $g(x) = h(y)$ as

$$\inf_{\Sigma, \theta \in \mathbb{R}} \left(\tfrac{1}{2}\Sigma^2 \tfrac{x}{g'(x)} - \alpha_q(\Sigma, \theta)x - \beta(\theta)g(x) + \gamma(\theta) \right) = 0, \ x \in (\underline{x}, \overline{x}),$$

$$g'(\underline{x}) = g'(\overline{x}) = 0 \text{ and } \int_{\underline{x}}^{\overline{x}} \frac{g'(x)}{x} \, dx = \log(\tfrac{1+\overline{\lambda}}{1-\underline{\lambda}}) = \overline{y} - \underline{y}.$$

Solving for g

The equation for g can be rewritten as

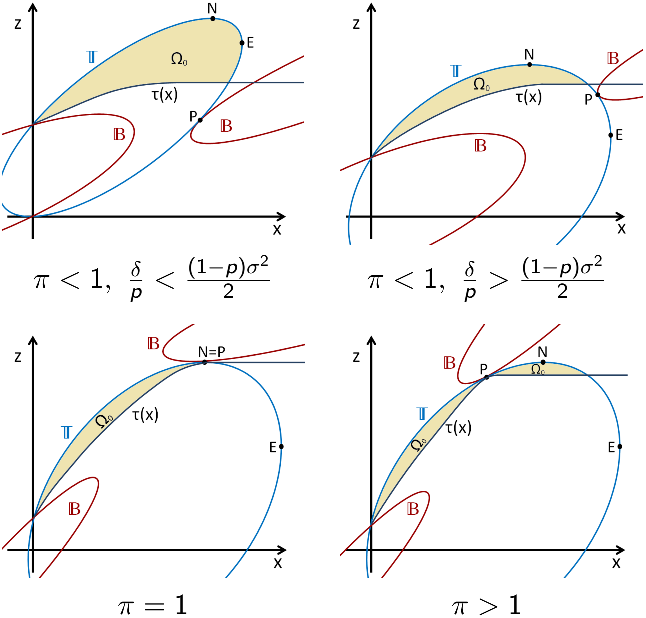
$$g' = \frac{P(x,g)}{Q(x,g)},$$

where P, Q are second order polynomials.
Idea of the solution:

- fix $(\alpha, g(\alpha)) \in \{P = 0\}$
- evolve the solution of the ODE until it meets again $\{P = 0\}$ at $x = \beta_\alpha$
- "move" α to meet the integral constraint

$$\int_\alpha^{\beta_\alpha} \frac{g'(x)}{x} dx = \overline{y} - \underline{y}.$$

For $\alpha \searrow 0$ the integral is ∞. The integral is decreasing in α. Then, choose $\underline{x} = \alpha, \overline{x} = \beta_\alpha$.



$\pi < 1,\ \frac{\delta}{p} < \frac{(1-p)\sigma^2}{2}$

$\pi < 1,\ \frac{\delta}{p} > \frac{(1-p)\sigma^2}{2}$

$\pi = 1$

$\pi > 1$

Figure: $0 < p < 1,\ \mu < G$.

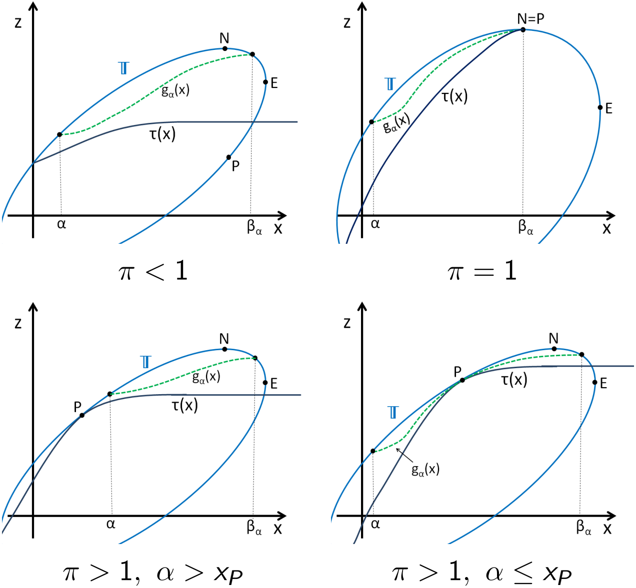


Figure: $0 < p < 1,\ \mu < G$

Possible singularity

A singularity appears when the NT region contains the axis $\eta_B = 0$.

Is there always a solution g?

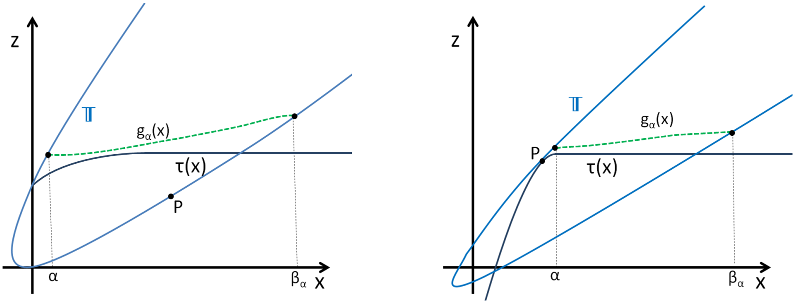
No, as, for some values of the parameters, we have

$$\lim_{\alpha \nearrow \infty} \int_{\alpha}^{\beta_\alpha} \frac{g'(x)}{x} dx = C(\mu, \sigma, \delta, p) > 0$$

This value can be computed explicitly!

- $\overline{y} - \underline{y} = \log(\frac{1+\overline{\lambda}}{1-\underline{\lambda}}) \leq C(\mu, \sigma, \delta, p)$ means the value function is infinite, and no solution g exists
- $\overline{y} - \underline{y} = \log(\frac{1+\overline{\lambda}}{1-\underline{\lambda}}) > C(\mu, \sigma, \delta, p)$ means the value function is finite, and there is a solution g

Full characterization of well posedness.



$\pi < 1$ $\qquad$ $\pi > 1, \ \alpha > x_P$

Figure: $0 < p < 1, \ G \leq \mu < A$

How do we solve the original problem?

Have $g(x)$. Change back the variable to $y = f(x)$ such that

$$f'(x) = -\frac{g'(x)}{x},$$

$$\underline{y} = f(\overline{x}), \underline{y} = f(\overline{x}).$$

With these notation, the value function of the game is

$$v(w, y) = \frac{w^p}{p} g(x)^{1-p}.$$

Also, whenever the state Y is at $y \leftrightarrow x$ it is optimal for the investor to hold $\pi(w, x) = \pi(x)$ proportion in the stock (driven by the factor Y), without transaction costs.

Solution of the original problem cont'd

- For fixed initial y, the investor will not buy/sell stocks invest expect when $Y = \overline{y}, \underline{y}$.
- The edges of the NT region have slopes $\pi(\underline{x}), \pi(\overline{x})$.

Choosing the intial y

Recall $Y_0 = y$, and denote $w(x) = \eta_B + \eta_S S_0 e^{f(x)}$. i.e. the total wealth at time $t = 0-$ if the consistent prices starts at $S_0 e^y$.
Choose y (or x) to minimize

$$v(w(x), x) = \frac{w(x)^p}{p} g(x)^{1-p}.$$

Denote again, $r(x) = \eta_S S_0 e^{f(x)} - \pi(x) w(x)$, i.e. the difference between how much the investor has in the stock at time $t = 0-$ and what will have a time $t = 0$ investing in the consistent price process S^y.
We can compute easily $\frac{d}{dx} v(w(x), x) = -A(x) r(x)$. Therefore

- if possible, choose x such that $r(x) = 0$ (start inside the NT region)
- if $r(x) > 0 \; \forall x$, then choose $x = \overline{x}$ (jump to the upper edge of the NT region)
- if $r(x) < 0 \; \forall x$, then choose $x = \underline{x}$ (jump to the lower edge of the NT region)

Main results

Theorem

(Well Posedness) Given the parameters $\mu, \sigma \in (0, \infty)$ *and the transaction costs* $\underline{\lambda} \in (0,1)$, $\overline{\lambda} > 0$, *the following statements are equivalent:*

(1) The problem is well posed, i.e

$$-\infty < u < \infty.$$

(2) The parameters of the model satisfy one of the following three conditions:

- $p \leq 0$,
- $0 < p < 1$ *and* $\mu < \sqrt{\frac{2\delta(1-p)\sigma^2}{p}}$,
- $0 < p < 1$, $\sqrt{\frac{2\delta(1-p)\sigma^2}{p}} \leq \mu < \frac{\delta}{p} + \frac{(1-p)\sigma^2}{2}$ *and*

$$C(\mu, \sigma, p, \delta) < \log\left(\frac{1+\overline{\lambda}}{1-\underline{\lambda}}\right),$$

where $C(\cdot,\cdot,\cdot,\cdot)$ *admits an explicit (closed-form) expression.*

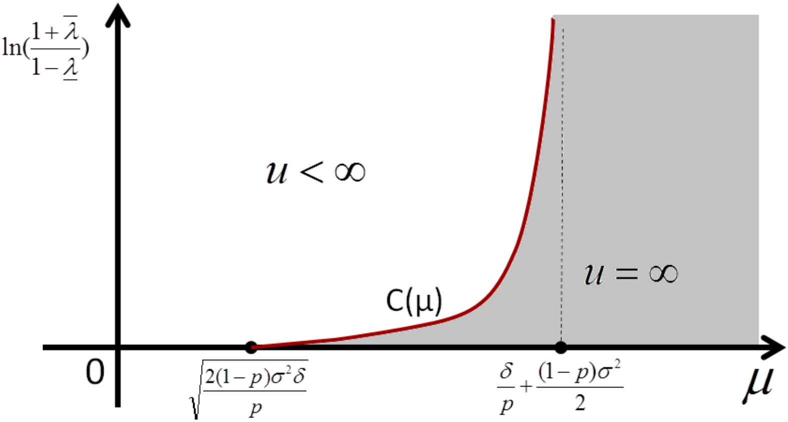


Figure: The well-posedness region.

Remark: generalizes Shreve and Soner, who had this condition only for $\sigma = 0$

Main results, cont'd

Theorem

If the problem is well posed, there exists a solution $g : [\underline{x}, \overline{x}] \to \mathbb{R}$ *and a shadow price* $\tilde{S}_t = S_t e^{Y_t}$*, that leads to the optimal solution of the original problem.*

Conclusions

We have a new way to solve the problem of optimal investment/consuption with transaction costs for power utilitites:

- self contained and direct (no dynamic programming principle)
- more elementary
- complete: treats all parameter values, explicitly characterize well posedness

Lays the foundation for expansion as power series of $\lambda^{\frac{1}{3}}$ of **any order**: see the paper of Jin Hyuk Choi. Closed form solution as a power series.

Future work

Represent general singular stochastic control problems as (absolutely continuous) games (with state constraints).