Shadow prices and well-posedness in the problem of optimal investment and consumption with transaction costs

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Objective

- revisit the classic problem of Davis and Norman using the so-called "shadow price" approach
- treat all possible values of parameters (and therefore, study the well posedness of the problem)
One risky asset (stock)

\[ dS_t = S_t(\mu \, dt + \sigma \, dB_t), \quad t \in [0, \infty) \text{ with } S_0 > 0. \]

Transaction costs: \( \lambda \in (0, 1) \) and \( \overline{\lambda} > 0 \):

- one gets only \( S_t = (1 - \lambda)S_t \) for one share of the stock
- pays \( \overline{S}_t = (1 + \overline{\lambda})S_t \) for it

Bond pays zero interest rate: \( r = 0 \).
Investment/consumption strategies

(Kallsen-Muhle Karbe notation)

- $\varphi^0_t$: number of bonds held at time $t$ (after the transaction, if any, took place)
- $\varphi_t$: number of stocks held at time $t$ (same as above)
- $c_t$: consumption rate

The processes $\varphi$ and $\varphi^0$ are right-continuous and of finite variation and $c$ is nonnegative and locally integrable.

Distinguish between the initial values $(\varphi^0_0, \varphi^0_0)$ and the values $(\varphi^0_0, \varphi_0)$ (after which the processes are right-continuous). Initial position

$$(\varphi^0_0, \varphi_0) = (\eta_B, \eta_S).$$

Remark: the initial jump appears even if there is no transaction cost.
Self-financing strategy \((\varphi^0, \varphi, c)\):

\[
\varphi^0_t = \varphi^0_{0^-} - \int_0^t \bar{S}_u d\varphi^+_u + \int_0^t S_u d\varphi^+_u - \int_0^t c_u \, du, \quad (1)
\]

where \(\varphi = \varphi^0_{0^-} + \varphi^+ - \varphi^-\) is the pathwise minimal (Hahn-Jordan) decomposition of \(\varphi\).

Remark: there is a possible jumps at time zero, as we assume that \(\varphi^+_{0^-} = \varphi^-_{0^-} = 0\).

Admissible strategy: can always be liquidated

\[
\varphi^0_t + \varphi^+_t S_t - \varphi^-_t \bar{S}_t \geq 0, \quad \forall t \geq 0^-.
\]
Optimal investment/consumption

For \( p \in (-\infty, 1) \), we consider the utility function
\[
U : [0, \infty) \to [-\infty, \infty)
\]
of the power (CRRA) type
\[
U(c) = \begin{cases} 
\frac{1}{p} c^p, & c \neq 0, p \neq 0 \\
\log(c), & c \neq 0, p = 0,
\end{cases}
\]
and
\[
U(0) = \begin{cases} 
0, & p > 0 \\
-\infty, & p \leq 0
\end{cases}
\]

Optimization problem:
\[
u = \sup_{(\phi^0, \phi, c)} \mathbb{E} \left[ \int_0^\infty e^{-\delta t} U(c_t) \, dt \right],
\]

where \( \delta > 0 \) stands for the (constant) impatience rate.
Literature

- Constantinides and Magill (76): heuristic solution
- Davis and Norman (90): analytic solution
- Shreve and Soner (94): removed some technical conditions, making use of viscosity theory. Key assumption: well posedness

\[ u < \infty \]

Conclusion: there exists a wedge around the Merton proportion such that

- jump to the boundary of the wedge
- do not trade inside the wedge
Shadow prices

Introduced by

- Jouini and Kallal (95)
- Lamberton, Pham and Schweizer (98)

1. Consistent price process: $S_t \leq \tilde{S}_t \leq \overline{S}_t$, for all $t \geq 0$. Investor with wealth $\eta_B + \eta_S \tilde{S}_0$ can certainly do better trading in $\tilde{S}$ without transaction costs, than trading in $S$ with transaction costs.

2. If one can find a consistent price process $\tilde{S}$ such that the corresponding optimal investment strategy (without transaction costs) trades as
   - buy stocks only when $\tilde{S} = \overline{S}$
   - sell stocks only when $\tilde{S} = S$

then such a trading strategy is actually optimal in the problem with transaction costs.

The price process $\tilde{S}$ is called shadow price.
Can this idea be used for the Davis and Norman problem?

Yes, in some cases

- Kallsen and Muhle Karbe, for $p = 0$. For fixed $\tilde{S}$ use the explicit solution of the log investor without transaction costs to find a shadow price.

- Herczegh and Prokaj (parallel to our work): use the solution in Shreve and Soner (the primal value function) to construct a shadow price for $p \neq 1$. The shadow price is constructed in terms of the gradient of the primal value function.

Both interpret the problem as a one dimensional free-boundary problem in terms of the variable

$$\log\left(\frac{\pi}{1 - \pi}\right).$$

Both require restriction on parameters: NT region is in the first quadrant (no singularity in our coming approach).
Our shadow price approach

Exploit the fact that the shadow price is a worst case scenario among consistent prices $S \leq \tilde{S} \leq \bar{S}$.

Parametrize consistent processes $\tilde{S}$ by $(\theta, \Sigma)$ such that

$$d\tilde{S}_t = \tilde{S}_t(\sigma + \Sigma_t) (dB_t + \theta_t \, dt), \quad \tilde{S}_0 = \tilde{s}_0.$$  

Pass to the logarithmic scale $Y_t = \log(\tilde{S}_t/S_t)$, whose dynamics is given by

$$dY_t = \alpha_0(\theta_t, \Sigma_t) \, dt + \Sigma_t \, dB_t,$$

$$Y_0 = y = \log\left(\frac{\tilde{s}_0}{S_0}\right).$$

on the natural domain $Y_t \in [\underline{y}, \overline{y}]$. Here,

$$\underline{y} = \log(1 - \lambda), \quad \overline{y} = \log(1 + \lambda)$$

and $\alpha_0(\Sigma, \theta) = \theta\sigma - \mu - \Sigma \left(\frac{1}{2}\Sigma + \sigma - \theta\right)$. 
Shadow prices as a game

Recall that, $\eta_B, \eta_S$ are fixed. Fix $y \in [\bar{y}, \underline{y}]$ and $\theta, \Sigma$ admissible.

1. Solve the optimal investment problem for an investor with initial wealth

$$w = \eta_B + \eta_S S_0 e^y$$

i.e. maximize over $\pi, c$ (proportion of investment in $\tilde{S}$ and rate of consumption) the expected discounted utility

$$E \left[ \int_0^\infty e^{-\delta t} U(c_t) \right].$$

2. Minimize first over $(\theta, \Sigma)$

3. Minimize over $y \in [\bar{y}, \underline{y}]$.

Summarize: up to the last minimization over $y$, we have a problem

$$v(w, y) := \inf_{\theta, \Sigma} \sup_{\pi, c} E \left[ \int_0^\infty e^{-\delta t} U(c_t) \right].$$
Game cont’d

We can write the Isaacs equation for the game with a two-dimensional state \((W, Y)\). The problem actually scales as

\[
v(w, y) = \frac{w^p}{p} h(y)^{1-p},
\]

and the Isaacs equation reduces to a one-dimensional equation for \(h(y), y \in [\bar{y}, \underline{y}]\).

Actually, from the duality theory for complete markets, we can easily see that

\[
h(y) = \inf_{(\Sigma, \theta) \in \mathcal{P}(y)} \mathbb{E} \left[ \int_0^\infty e^{-\delta t} V \left( e^{\delta t} \mathcal{E}(\theta \cdot B)_t \right) dt \right]
\]

The reduced Isaacs equation for \(h(y)\) is actually an HJB (this is what we use in the paper).

Remark: the sup and inf separate in the game.
The one-dimensional HJB

\[
\inf_{\Sigma, \theta} \left( \frac{1}{2} \Sigma^2 h''(y) + \alpha_q(\Sigma, \theta) h'(y) - \beta(\theta) h(y) + \gamma(\theta) \right) = 0, \quad y \in (\underline{y}, \bar{y})
\]

\[w''(\underline{y}) = w''(\bar{y}) = +\infty.\]

\[q = \frac{p}{1-p}, \quad \alpha_q(\Sigma, \theta) = \theta \sigma - \mu - \Sigma \left( \frac{1}{2} \Sigma + \sigma - \theta(1 + q) \right),\]

\[\beta(\theta) = (1 + q) \left( \delta - \frac{1}{2} q \theta^2 \right), \quad \text{and} \quad \gamma(\theta) = \begin{cases} 
\frac{1}{2} \theta^2, & p = 0, \\
\text{sgn}(p), & p \neq 0.
\end{cases}\]  \hspace{1cm} (2)

The boundary conditions ensure the state constraint \( Y \in [\underline{y}, \bar{y}] \).
Change of variable/order reduction

The HJB has the form

\[ H(h'', h', h) = 0, \quad y \in [\bar{y}, \underline{y}] . \]

Expect \( h' \) to be

- increasing
- negative

Change the variable to \( x = -h'(y) \). The (fixed) boundaries \( y, \bar{y} \) change to

\[ \underline{x} = -h'(\bar{y}), \bar{x} = -h'(\underline{y}) . \]

Rewrite the HJB by \( g(x) = h(y) \) as

\[
\inf_{\Sigma, \theta \in \mathbb{R}} \left( \frac{1}{2} \Sigma^2 \frac{x}{g'(x)} - \alpha_q(\Sigma, \theta)x - \beta(\theta)g(x) + \gamma(\theta) \right) = 0, \quad x \in (\underline{x}, \bar{x}),
\]

\[ g'(\underline{x}) = g'(\bar{x}) = 0 \text{ and } \int_{\underline{x}}^{\bar{x}} \frac{g'(x)}{x} \, dx = \log\left( \frac{1+\lambda}{1-\lambda} \right) = \bar{y} - \underline{y} . \]
Solving for $g$

The equation for $g$ can be rewritten as

$$g' = \frac{P(x, g)}{Q(x, g)},$$

where $P$, $Q$ are second order polynomials.

Idea of the solution:

- fix $(\alpha, g(\alpha)) \in \{P = 0\}$
- evolve the solution of the ODE until it meets again $\{P = 0\}$ at $x = \beta_\alpha$
- ”move” $\alpha$ to meet the integral constraint

$$\int_{\alpha}^{\beta_\alpha} \frac{g'(x)}{x} \, dx = \bar{y} - y.$$

For $\alpha \searrow 0$ the integral is $\infty$. The integral is decreasing in $\alpha$. Then, choose $\underline{x} = \alpha, \overline{x} = \beta_\alpha$. 
\[ \pi < 1, \quad \frac{\delta}{p} < \frac{(1-p)\sigma^2}{2} \]

\[ \pi < 1, \quad \frac{\delta}{p} > \frac{(1-p)\sigma^2}{2} \]

\[ \pi = 1 \]

\[ \pi > 1 \]

\textbf{Figure:} 0 < p < 1, \mu < G.
\[ \pi < 1 \]

\[ \pi = 1 \]

\[ \pi > 1, \; \alpha > \chi_P \]

\[ \pi > 1, \; \alpha \leq \chi_P \]

\text{Figure: } 0 < p < 1, \; \mu < G
Possible singularity

A singularity appears when the NT region contains the axis $\eta_B = 0$. 
Is there always a solution \( g \)?

No, as, for some values of the parameters, we have

\[
\lim_{\alpha \to \infty} \int_{\alpha}^{\beta} \frac{g'(x)}{x} \, dx = C(\mu, \sigma, \delta, p) > 0
\]

This value can be computed explicitly!

\( \bar{y} - y = \log(\frac{1+\lambda}{1-\lambda}) \leq C(\mu, \sigma, \delta, p) \) means the value function is infinite, and no solution \( g \) exists

\( \bar{y} - y = \log(\frac{1+\lambda}{1-\lambda}) > C(\mu, \sigma, \delta, p) \) means the value function is finite, and there is a solution \( g \)

Full characterization of well posedness.
Figure: $0 < p < 1, \ G \leq \mu < A$
How do we solve the original problem?

Have \( g(x) \). Change back the variable to \( y = f(x) \) such that

\[
f'(x) = -\frac{g'(x)}{x},
\]

\( y = f(x), \quad \underline{y} = f(\underline{x}). \)

With these notation, the value function of the game is

\[
v(w, y) = \frac{wp}{p} g(x)^{1-p}.
\]

Also, whenever the state \( Y \) is at \( y \leftrightarrow x \) it is optimal for the investor to hold \( \pi(w, x) = \pi(x) \) proportion in the stock (driven by the factor \( Y \)), without transaction costs.
Solution of the original problem cont’d

- For fixed initial $y$, the investor will not buy/sell stocks invest expect when $Y = \bar{y}, \underline{y}$.
- The edges of the NT region have slopes $\pi(\underline{x}), \pi(\bar{x})$. 
Choosing the initial $y$

Recall $Y_0 = y$, and denote $w(x) = \eta_B + \eta_S S_0 e^{f(x)}$. i.e. the total wealth at time $t = 0$—if the consistent prices starts at $S_0 e^y$.

Choose $y$ (or $x$) to minimize

$$v(w(x), x) = \frac{w(x)^p}{p} g(x)^{1-p}.$$

Denote again, $r(x) = \eta_S S_0 e^{f(x)} - \pi(x) w(x)$, i.e. the difference between how much the investor has in the stock at time $t = 0$—and what will have at time $t = 0$ investing in the consistent price process $S^y$.

We can compute easily $\frac{d}{dx} v(w(x), x) = -A(x) r(x)$. Therefore

- if possible, choose $x$ such that $r(x) = 0$ (start inside the NT region)
- if $r(x) > 0 \forall x$, then choose $x = \overline{x}$ (jump to the upper edge of the NT region)
- if $r(x) < 0 \forall x$, then choose $x = \underline{x}$ (jump to the lower edge of the NT region)
Main results

Theorem

(Well Posedness) Given the parameters $\mu, \sigma \in (0, \infty)$ and the transaction costs $\lambda \in (0, 1)$, $\lambda > 0$, the following statements are equivalent:

(1) The problem is well posed, i.e

$$-\infty < u < \infty.$$  

(2) The parameters of the model satisfy one of the following three conditions:

- $p \leq 0$,
- $0 < p < 1$ and $\mu < \sqrt{\frac{2\delta(1-p)\sigma^2}{p}}$,
- $0 < p < 1$, $\sqrt{\frac{2\delta(1-p)\sigma^2}{p}} \leq \mu < \frac{\delta}{p} + \frac{(1-p)\sigma^2}{2}$ and

$$C(\mu, \sigma, p, \delta) < \log\left(\frac{1+\lambda}{1-\lambda}\right),$$

where $C(\cdot, \cdot, \cdot, \cdot, \cdot)$ admits an explicit (closed-form) expression.
Remark: generalizes Shreve and Soner, who had this condition only for \( \sigma = 0 \)
Main results, cont’d

**Theorem**

*If the problem is well posed, there exists a solution* \( g: [\bar{x}, \bar{x}] \rightarrow \mathbb{R} \) 
*and a shadow price* \( \tilde{S}_t = S_t e^{Y_t} \), *that leads to the optimal solution of the original problem.*
Conclusions

We have a new way to solve the problem of optimal investment/consumption with transaction costs for power utilities:

▶ self contained and direct (no dynamic programming principle)
▶ more elementary
▶ complete: treats all parameter values, explicitly characterize well posedness

Lays the foundation for expansion as power series of $\lambda^{1/3}$ of any order: see the paper of Jin Hyuk Choi. Closed form solution as a power series.
Future work

Represent general singular stochastic control problems as (absolutely continuous) games (with state constraints).