APPENDIX: EXPLANATION OF THEOREM 5.2 OF [LPSZ19]

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Let $G = \mathrm{GSp}(4)$, with Siegel parabolic P_S and block diagonal Levi subgroup M_S, then $M_S \simeq (\mathrm{GL}_2 \times \mathrm{GL}_1)/\Delta\mu_2$. Let T be the diagonal maximal torus of G, then we label the characters of T as

$$(r_1, r_2; c) : \mathrm{diag}(st_1, st_2, st_2^{-1}, st_1^{-1}) \mapsto t_1^{r_1} t_2^{r_2} s^c$$

where $r_1, r_2, c \in \mathbf{Z}$, $r_1 + r_2 \equiv c \pmod 2$. The roots of G are $\{(\pm 2, 0; 0), (0, \pm 2; 0), (\pm 1, \pm 1; 0)\}$, with a choice of positive simple roots being $\{(0, 2; 0), (1, -1; 0)\}$. The corresponding simple weights are $\{(1, 1; 0), (1, 0; 0)\}$. A character $(r_1, r_2; c)$ is dominant for M_S exactly if $r_1 \geq r_2$, and it is dominant for G exactly if $r_1 \geq r_2 \geq 0$. The Weyl orbit of $(r_1, r_2; c)$ is $\{(\pm r_1, \pm r_2; c), (\pm r_2, \pm r_1; c)\}$. Finally, $\rho = (2, 1; 0)$.

Let K_∞ be the maximal compact subgroup of $G(\mathbf{R})_+$ fixing $iI_2 \in \mathcal{H}_2$. It is isomorphic to $\mathbf{R}^\times \cdot U(2)$. We have a decomposition $\mathfrak{g}_\mathbf{C} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$, where $\mathfrak{k} = \mathrm{Lie}(K_\infty)_\mathbf{C}$, and $\mathfrak{p}^\pm$ can be viewed as the space of holomorphic or antiholomorphic derivations at $iI_2 \in \mathcal{H}_2$. Therefore, $\dim \mathfrak{k} = 5$ and $\dim \mathfrak{p}^\pm = 3$. Let P be the parabolic subgroup associated to $\mathfrak{p} = \mathfrak{k} \oplus \mathfrak{p}^-$, then P is conjugate to P_S, and this conjugation identifies $(K_\infty)_\mathbf{C}$ with M_S. The action of K_∞ on $\mathfrak{p}^-$ has highest weight $(0, -2; 0)$.

Given a dominant weight λ, let V_λ be finite dimensional algebraic representation of M_S with highest weight λ. This induces a coherent sheaf $[V_\lambda]$ on $X_{G,\mathbf{C}}$. Let $\Pi = \Pi_f \otimes \Pi_\infty$ be a generic cuspidal automorphic representation on G, and further suppose that Π_∞ is a discrete series representation with Harish-Chandra parameter $\chi = (r_1, r_2; c) + \rho$, where $r_1 \geq r_2 \geq 0$. We want to understand the Π_f-component of $H^*(X_{G,\mathbf{C}}, [V_\lambda])$. It follows from the main theorems of [Har90] (for sufficiently regular weights) and [Su19] (in general) that

$$H^*(X_{G,\mathbf{C}}, [V_\lambda]) \simeq H^*(\mathfrak{p}, K_\infty; \mathcal{A}(G) \otimes V_\lambda) = H^*(\mathrm{Hom}_{K_\infty}(\wedge^\bullet \mathfrak{p}^- \otimes V_\lambda^\vee, \mathcal{A}(G)))$$

Taking Π_f-components, we see that we need to understand automorphic representations on G whose finite part is Π_f. First, since Π is not a CAP form [AS06], Π_f does not contribute to the continuous spectrum. Using Arthur's classification, the infinity component must be in the same L-packet as Π_∞, so it is in the discrete series and moreover has infinitesimal character χ.

The discrete series in an L-packet is classified by the coset space $W(G, S)\backslash W(G_\mathbf{C}, S)$, where S is an elliptic maximal torus. In our case, this gives two representations Π_∞^H and Π_∞^W, where H stands for holomorphic and W stands for generic. The local component group $\mathcal{S}_{\varphi_\infty}$ is $\mathbf{Z}/2\mathbf{Z}$. The local pairing attaches the trivial character to Π_∞^W and the non-trivial character to Π_∞^H, following the procedure described in [BR94, Section 5.3]. Since our parameter is tempered, Arthur's multiplicity formula gives

$$m(\Pi) = \frac{1}{|\mathcal{S}_\varphi|} \sum_{s \in \mathcal{S}_\varphi} \langle s, \Pi \rangle$$

In the stable case, the global component group is trivial, so $m(\Pi_f \otimes \Pi_\infty^\square) = 1$ for $\square \in \{H, W\}$. In the Yoshida case, both $\mathcal{S}_\varphi$ and $\mathcal{S}_{\varphi_\infty}$ are $\mathbf{Z}/2\mathbf{Z}$. At the finite places, since Π_v is generic, the local pairing is 1. Therefore, the global character $\langle -, \Pi \rangle$ is trivial for Π_∞^W and non-trivial for Π_∞^H. So in the Yoshida case, only Π_∞^W can be the infinity component.

When restricted to the $(\mathfrak{p}, K_\infty)$-module, both Π_∞^H and Π_∞^W decomposes into two irreducible representations. Their $(\mathfrak{p}, K_\infty)$-cohomology is computed by [Har90, Theorem 4.6.2] (note that there is an additional dual in its statement). With respect to the chosen compact maximal torus, the non-compact roots are $\pm(1, -1; 0)$. We have the following four cases:

Holomorphic: $\chi = (r_1 + 2, r_2 + 1; c)$, $\lambda = (-r_2 - 3, -r_1 - 3; -c)$, $q = 0$
Generic: $\chi = (r_1 + 2, -r_2 - 1; c)$, $\lambda = (r_2 - 1, -r_1 - 3; -c)$, $q = 1$
Anti-generic: $\chi = (r_2 + 1, -r_1 - 2; c)$, $\lambda = (r_1, -r_2 - 3; -c)$, $q = 2$
Anti-holomorphic: $\chi = (-r_2 - 1, -r_1 - 2; c)$, $\lambda = (r_1, r_2; -c)$, $q = 3$

The proof is roughly as follows: on the cochain complex $\mathrm{Hom}_{K_\infty}(\wedge^\bullet \mathfrak{p}^- \otimes V_\lambda^\vee, \Pi_\infty)$, one can define the Laplacian operator $\Delta = dd^* + d^*d$ and show that it acts as the scalar $\langle \lambda+\rho, \lambda+\rho\rangle - \langle \chi, \chi\rangle$. If this is non-zero, then the complex is exact, so there is no $(\mathfrak{p}, K_\infty)$-cohomology. Otherwise, all differentials are trivial, so we are left with a K_∞-type computation, which can be done by hand in our case. Also see [Sch17] for useful pictures. The results here are off from [LPSZ19] by a dual, probably relating to normalization issues. We now comment on the central character. For the analytic normalization, we have $c = 0$ to make the central character of Π finite order. But for the corresponding V_λ to be algebraic, we must have $c \equiv r_1 + r_2 \pmod 2$, so we need to twist Π by the character $\|\nu\|^{-\frac{r_1+r_2}{2}}$ to have algebraicity.

To get the theorem for the subcanonical extension, first note that Serre duality gives a perfect pairing

$$H^{3-i}(X_{G,\mathbf{Q}}, [V_\lambda]) \times H^i(X_{G,\mathbf{Q}}, [V_\lambda^\vee](-D)) \to \mathbf{Q}$$

This is also $G(\mathbb{A}_f)$-equivariant. Applying the previous argument to $\Pi^\vee$ instead of Π shows that the multiplicity statements in the theorem for the subcanonical extension as well. Since Π is cuspidal, the inclusion of $H^i(X_{G,\mathbf{Q}}, [V_\lambda](-D))$ into $H^i(X_{G,\mathbf{Q}}, [V_\lambda])$ is surjective on the Π_f-isotypic components. This combined with the multiplicity statements prove the isomorphism.

References

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