Determinant (Computations)

Frobenius normal form

Given real numbers $a_0, a_1, \cdots, a_{n-1}$, let $A_{a_0,\cdots,a_{n-1}}$ be the $n \times n$ matrix

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & & \ddots & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

We want to compute its characteristic polynomial (cf. Chapter 7), which is defined to be

$$p(\lambda) = \det(\lambda I_{n+1} - A_{a_0,\cdots,a_{n-1}})$$

Expand along the first row:

$$\det \begin{bmatrix} \lambda & 0 & \cdots & 0 & a_0 \\ -1 & \lambda & \cdots & 0 & a_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & & \ddots & \lambda & a_{n-2} \\ 0 & 0 & \cdots & -1 & a_{n-1} \end{bmatrix} = \lambda \det \begin{bmatrix} \lambda & 0 & \cdots & 0 & a_1 \\ -1 & \lambda & \cdots & 0 & a_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & & \ddots & \lambda & a_{n-2} \\ 0 & 0 & \cdots & -1 & a_{n-1} \end{bmatrix} + (-1)^{1+n)} a_0 \det \begin{bmatrix} -1 & \lambda & 0 & \cdots & 0 \\ 0 & -1 & \lambda & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & -1 & \lambda \\ 0 & & \cdots & 0 & -1 \end{bmatrix}$$
$$= \lambda \det(\lambda I_n - A_{a_1,\cdots,a_{n-1}}) + a_0$$

since the second matrix is an $(n-1) \times (n-1)$ upper triangular matrix whose diagonal entries are all -1. We can recursively expand this to get[1]

$$\det(\lambda I_n - A_{a_0,\cdots,a_n}) = a_0 + \lambda(a_1 + \lambda(a_2 + \cdots)) = a_0 + a_1\lambda + \cdots a_{n-1}\lambda^{n-1} + \lambda^n$$

This shows that we can find matrices with any given characteristic polynomial, and more over, the entries of the matrix only involve the coefficients, and not the roots.

TST matrices

Let a, b be two real numbers. Consider the $n \times n$ matrix

$$D^{(n)}_{a,b} = \begin{bmatrix} a & b & 0 & \cdots & 0 \\ b & a & b & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & b & a & b \\ 0 & \cdots & 0 & b & a \end{bmatrix}$$

This matrix is symmetric, tridiagonal (entries not within one of the main diagonal are zero), and Toeplitz (the entries are equal along a shifted diagonal). We will compute its determinant.

First observe the block matrix structure

$$D^{(n)}_{a,b} = \begin{bmatrix} a & b & 0 & \cdots & 0 \\ b & & & & \\ 0 & & & & \\ \vdots & & D^{(n-1)}_{a,b} & & \\ 0 & & & & \end{bmatrix} = \begin{bmatrix} a & b & 0 & \cdots & 0 \\ b & a & b & \cdots & 0 \\ 0 & b & & & \\ \vdots & \vdots & & D^{(n-2)}_{a,b} & \\ 0 & 0 & & & \end{bmatrix}$$

[1] This is also the fast polynomial evaluation algorithm, using only n multiplications

Expand along the first column:

$$\det D_{a,b}^{(n)} = a \det D_{(a,b)}^{(n-1)} - b \det \begin{bmatrix} b & 0 & \cdots & 0 \\ b & & & \\ \vdots & & D_{a,b}^{(n-2)} & \\ 0 & & & \end{bmatrix}$$

For the second matrix, we can expand along the first row. The end result is

$$\det D_{a,b}^{(n)} = a \det D_{(a,b)}^{(n-1)} - b^2 \det D_{(a,b)}^{(n-2)}$$

This is a linear recurrence relation of the type we will study in Chapter 7. The initial conditions are

$$\det D_{a,b}^{(1)} = a, \quad \det D_{a,b}^{(2)} = a^2 - b^2$$

For now, we will do an example.

Example. Let $a = 2$ and $b = 1$, then the above relation reads

$$\det D_{2,1}^{(n)} = 2 \det D_{(a,b)}^{(n-1)} - \det D_{(a,b)}^{(n-2)}, \ \det D_{a,b}^{(1)} = 2, \ \det D_{a,b}^{(2)} = 3$$

So for example, $\det D_{a,b}^{(3)} = 2 \times 3 - 2 = 4$. It is easy to see that the pattern continues, so $\det D_{2,1}^{(n)} = n + 1$. We will computer the eigenvalues of $D_{a,b}^{(n)}$ in Chapter 7. That computation combined with this formula will lead to an identity

$$\cos \frac{\pi}{2n} \cos \frac{2\pi}{2n} \cdots \cos \frac{(n-1)\pi}{2n} = \frac{\sqrt{n}}{2^{2n-2}}$$

Adjugate matrix

Let A be an $n \times n$ matrix, then its adjugate[2] is the $n \times n$ matrix defined by

$$\operatorname{adj}(A) := \left((-1)^{i+j} \det \hat{A}_{ji}\right)_{1 \le i,j \le n}$$

This satisfies the relation

$$\operatorname{adj}(A)A = A \operatorname{adj}(A) = \det(A) I_n$$

This is just a really compact way of stating the row expansion and column expansion formulae. We will see this in an example. From this, we get that A is invertible if and only if $\det(A) = 0$. We also get the formula for the inverse of A when it is invertible.

Example. Let A be the 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

then its adjugate matrix is

$$\operatorname{adj}(A) = \begin{bmatrix} \left|\begin{smallmatrix} e & f \\ h & i \end{smallmatrix}\right| & -\left|\begin{smallmatrix} b & c \\ h & i \end{smallmatrix}\right| & \left|\begin{smallmatrix} b & c \\ e & f \end{smallmatrix}\right| \\ -\left|\begin{smallmatrix} d & f \\ g & i \end{smallmatrix}\right| & \left|\begin{smallmatrix} a & c \\ g & i \end{smallmatrix}\right| & -\left|\begin{smallmatrix} a & c \\ d & f \end{smallmatrix}\right| \\ \left|\begin{smallmatrix} d & e \\ g & h \end{smallmatrix}\right| & -\left|\begin{smallmatrix} a & b \\ g & h \end{smallmatrix}\right| & \left|\begin{smallmatrix} a & b \\ d & e \end{smallmatrix}\right| \end{bmatrix}$$

So for example, entry $(1, 1)$ of $\operatorname{adj}(A)A$ is

$$a\left|\begin{smallmatrix} e & f \\ h & i \end{smallmatrix}\right| - d\left|\begin{smallmatrix} b & c \\ h & i \end{smallmatrix}\right| + g\left|\begin{smallmatrix} b & c \\ e & f \end{smallmatrix}\right| = \det(A)$$

Entry $(1, 2)$ of $A \operatorname{adj}(A)$ is

$$-a\left|\begin{smallmatrix} b & c \\ h & i \end{smallmatrix}\right| + b\left|\begin{smallmatrix} a & c \\ g & i \end{smallmatrix}\right| - c\left|\begin{smallmatrix} a & b \\ g & h \end{smallmatrix}\right| = \det \begin{bmatrix} a & b & c \\ a & b & c \\ g & h & i \end{bmatrix} = 0$$

where for the first equality, we were doing expansion along the second row.

[2] The textbook uses "classical adjoint"