Fredholm alternative

Since there was some confusion in lecture, here is the statement again. This is non-examinable in the sense that it is not in the textbook, but you may be asked some True/False questions in the exams based on this. Also, it is good to know.

Theorem. Let A be an $n \times n$ square matrix. Exactly one of the following two statements hold:

1. For every $\vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has a solution, xor
2. The equation $A\vec{x} = \vec{0}$ has at least one solution which is not $\vec{0}$.

Moreover,

- In the first case, $A\vec{x} = \vec{b}$ has exactly one solution for every $\vec{b} \in \mathbb{R}^n$.
- In the second case, $A\vec{x} = \vec{0}$ has infinitely many solutions.

After Chapter 3, I may refer back to this and give a nicer statement. In the meantime, here is a proof. It is not the shortest one, but it is a good review of things we did in lecture.

Proof. Let $r = \operatorname{rank}(A)$, then either $r = n$ xor $r < n$. In the first case, by the statement I labelled (b') in lecture, the system $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^n$, so statement (1) holds.

In the second case, the statement I labelled (c1') shows that for each $\vec{b} \in \mathbb{R}^n$, the system $A\vec{x} = \vec{b}$ is either inconsistent or has infinitely many solutions. In particular, let $\vec{b} = \vec{0}$. Observe that $A\vec{0} = \vec{0}$, so the system $A\vec{x} = \vec{0}$ is consistent. Therefore, it has infinitely many solutions, and statement (2) holds.

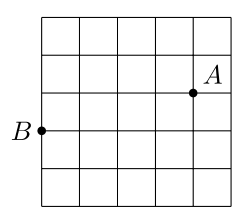
Moreover, suppose statement (1) holds, then the rref of A must be the identity matrix

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{n \times n}$$

Otherwise, we would have a zero row, and setting the corresponding row in $\vec{b}$ to 1 will give an inconsistent system. It follows that $A\vec{x} = \vec{0}$ has the unique solution $\vec{0}$. This shows that statements (1) and (2) cannot simultaneously hold. □

As an application, we consider the problem introduced in Lecture 1, which is officially known as the discrete Dirichlet boundary problem.

Theorem. Consider the following wire grid



Suppose the temperature at each interior point is the average of the temperatures at its four neighbours. Given any temperature distribution on the boundary, the temperatures in the interior are uniquely defined.

Proof. Let n be the number of lines on each side, so $n = 6$ in the above diagram. Recall that we set up a system of n^2 equations in n^2 unknowns. For an interior point, it corresponds to an equation of the form

$$4A - (N_1 + N_2 + N_3 + N_4) = 0$$

and for a boundary point, it is simply

$$B = (\text{given value})$$

The conclusion follows from statement (1) of the Fredholm alternative, so we show that statement (2) of the Fredholm alternative does not hold. The negation of statement (2) is that $A\vec{x} = \vec{0}$ has exactly one solution. Physically, we are setting the temperatures at the boundary to be 0.

Now, the point with the maximal temperature must occur at the boundary, since otherwise it is an average of its four neighbours. Similarly, this holds for the minimal temperature. But the maximal and minimal temperatures of the boundary are both 0, so the temperatures inside must all be 0. Therefore, $A\vec{x} = \vec{0}$ has a unique solution, disproving statement (2) of the Fredholm alternative. □

Note that the proof is quite general and does not really use much about the exact nature of the grid. In addition, the Fredholm alternative also holds for "infinite dimensional systems", interpreted correctly (this was a small part of Fredholm's work). In this form, it is used extensively in research, for example to prove theorems such as

Theorem. Given a smooth flat heat conducting plate. Suppose its temperature at boundary is specified and the plate is in thermal equilibrium, then there exists a unique interior temperature profile.

Mathematically, this is equivalent to finding the electric potential inside a conductor. Both are described by the Laplace equation, and the grid problem is a discretization of this problem.

In some cases, we can actually write down the solution to the continuous problem as an infinite series, but the series converges very slowly if the boundary temperature is discontinuous (for example, one side is 600K while the other three sides are 300K). In practice, people find the interior temperature by first discretizing it and then solving the linear system. Gauss–Jordan elimination is $O((n^2)^3) = O(n^6)$ time, which is not practical. When we do eigenvalues, I may talk about the discrete Fourier transform, which can solve it in $O(n^2 \log n)$ time. Of course, the $\log n$ came from Fast Fourier Transform.