Problem 1. Amitesh Q2: basis for image, kernel. Left vs. right inverse.
Problem 2.

(a) Let $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find all matrices C such that $CJ = JC$. Explain your answer geometrically.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} -y \\ x \end{bmatrix}$. Find the matrix of T in the basis

$$\mathfrak{B} = \left\{ \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

(c) Let A be the matrix you computed in part (b). Using the result of (a) or otherwise, find all invertible matrices B such that $BAB^{-1} = A$.

Problem 3. The 3×3 *Vandermonde matrix* has the following form

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$$

You are given that if a, b, c are mutually distinct, then A is invertible.

(a) Find all values of k such that the equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 5 & 12 & \sqrt{2} \\ 25 & 144 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & k \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 6 \\ 4 & 16 & 36 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 10 \\ \sqrt{3} \end{bmatrix}$$

has a unique solution.

(b) Explain why A is not invertible if $a = b$.

(c) Let $a = b = 1$, $c = 2$. Find a basis for $\mathrm{Im}(A)$.

(d) Find a basis of $\mathbb{R}^3$ which contains the vectors you wrote down in the previous part of the question.

Problem 4.

(a) Rotation CCW $2\pi/3$ and CW $2\pi/3$ similar?

(b) AB invertible $\implies \ker(A) = 0$.

(c) $Ax = b$ question: rank-nullity.

Problem 5. Let A be the 3×3 matrix with only ones

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(a) Find a basis for $\ker(A)$.

(b) Find an invertible matrix S such that

$$S^{-1}AS = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

[Hint: you already found the first two columns of S]

(c) In closed form, evaluate A^{2020}.

(d) Let B be the matrix

$$B = \begin{bmatrix} 1 & 1 & \lambda \\ 1 & \lambda & 1 \\ \lambda & 1 & 1 \end{bmatrix}$$

Find all values of λ such that B is not invertible. For each λ you found, find the rank of B.