Local Harmonic Analysis and Euler Systems

(joint work with Li Cai and Yangyu Fan)

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Diagonal cycle

E/F CM extension
$W \subseteq V$ Hermitian spaces of dimensions n and $n+1$.
$\rightsquigarrow$ Unitary groups $\mathbf{H} = \mathrm{U}(W)$, $\mathbf{G} = \mathrm{U}(W) \times \mathrm{U}(V)$

$$\mathbf{H} \hookrightarrow \mathbf{G}$$

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$\rightsquigarrow$ Embedding of Shimura varieties

$$\mathrm{Sh}_{\mathbf{H}}(K_{\mathbf{H}}) \hookrightarrow \mathrm{Sh}_{\mathbf{G}}(K_{\mathbf{G}})$$

$\rightsquigarrow$ Diagonal cycle

$$\triangle = [\mathrm{Sh}_{\mathbf{H}}] \in \mathrm{CH}^*(\mathrm{Sh}_{\mathbf{G}}(K_{\mathbf{G}}))$$

GGP setting

Suppose W and V are "nearly definite", so their signatures are

- $(1, n-1), (1, n)$ at one fixed archimedean place.
- $(0, n), (0, n+1)$ at other archimedean places.

Then $\dim \mathrm{Sh}_{\mathbf{H}} = n-1$, $\dim \mathrm{Sh}_{\mathbf{G}} = 2n-1$, so obtain

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Let p be a prime, then can take the p-adic étale realization

$$\triangle_p \in \mathrm{H}^{2n}_{\mathrm{cont}}(\mathrm{Sh}_{\mathbf{G}}(K_{\mathbf{G}}), \mathbb{Z}_p(n))$$

Can also replace coefficient $\mathbb{Z}_p$ by a $\mathbb{Z}_p$-local system $\mathbb{L}$, subject to branching law conditions.

Main theorem

Theorem (L.–Skinner)

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Using usual techniques of sign projector and Abel–Jacobi map, this produces Euler systems for certain Rankin–Selberg motives.

$+$Jetchev–Nekovář–Skinner $\implies$ progress towards rank 1 cases of Bloch–Kato conjecture.

New feature

We work integrally already at the motivic level, so our Bloch–Kato result applies to *all primes p* in many cases.

Tame part

Key step: for all but finitely many F-places ℓ which splits in E, we construct classes

$$\triangle_p^{(\ell)} \in \mathrm{H}^{2n}_{\mathrm{cont}}(\mathrm{Sh}_{\mathbf{G}}(K_{\mathbf{G}})_{/E[\ell]}, \mathbb{L}(n))$$

such that

$$\mathrm{Tr}_E^{E[\ell]} \triangle_p^{(\ell)} = \mathscr{L} \cdot \triangle_p$$

where $\mathscr{L}$ is the Hecke operator on $\mathbf{G}(F_\ell)$ whose Satake transform is

$$\hat{\mathscr{L}} = \prod_{1 \le i \le n} \prod_{1 \le j \le n+1} \left(1 - \mathbf{N}\ell^{-\frac{1}{2}} Z_i W_j\right)$$

i.e. inverse of the local L-factor.

Proof in L.–Skinner

Use twisting element

$$\delta_1' = \sum_{\eta \in \mathcal{N}_n'} \mu_H(K_H^{\varphi})^{-1}(-1)^{s(\eta)}(\ell - 1)^n \mathbf{1}[(1,\eta)K_G \times J_1]$$

Prove “local Birch lemma” by explicit matrix computations.

Goal of the talk

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1. For $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$, construct $\mathbf{G}(\mathbb{A}^{p\infty})$-equivariant map

$$\Theta^{p\infty} : C_c^{\infty}(\mathbf{X}(\mathbb{A}^{p\infty}), \mathbb{Z}_p) \to \mathrm{H}^{2n}_{\mathrm{cont}}(\mathrm{Sh}_{\mathbf{G}}, \mathbb{L}(n))$$

2. Describe the image of a certain trace map on $C_c^{\infty}(\mathbf{X}(F_\ell), \mathbb{Z}_p)$
3. Show that the function $\mathscr{L} \cdot \mathbf{1}[\mathbf{X}(\mathcal{O}_\ell)]$ lands in the image using relative Satake transform.

All steps should be part of a broader picture.

Twisting formalism

Theorem (Loeffler–Skinner–Zerbes)

There is a $\mathbf{G}(\mathbb{A}^{p\infty})$-equivariant map

$$C_c^\infty(\mathbf{G}(\mathbb{A}^{p\infty}), \mathbb{Z}_p) \to \mathrm{H}^{2n}_{\mathrm{cont}}(\mathrm{Sh}_{\mathbf{G}}, \mathbb{L} \otimes \mathbb{Q}_p(n))$$

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Integrality issue

For this to be well-defined, need to multiply by volume terms, destroying integrality.

Miracle?

We have the following commutative diagram

$$\begin{array}{ccc} C_c^\infty(\mathbf{G}(\mathbb{A}^{p\infty}), \mathbb{Z}_p) & & \\ \Big\downarrow \text{coinvariant} & \overset{\text{LSZ}}{\searrow} & \\ C_c^\infty(\mathbf{X}(\mathbb{A}^{p\infty}), \mathbb{Q}_p) & \longrightarrow & \mathrm{H}^{2n}_{\mathrm{cont}}(\mathrm{Sh}_{\mathbf{G}}, \mathbb{L} \otimes \mathbb{Q}_p(n)) \end{array}$$

The coinvariant map and the LSZ-map both destroy integrality, but in the *same way*.

Proposition (Cai–Fan–L., used in L.–Skinner)

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Field extension

Recall that ℓ is a place in F which splits in E. Introduce level structures

$$K = \mathbf{G}(\mathcal{O}_\ell) = \mathrm{GL}_n(\mathcal{O}_\ell) \times \mathrm{GL}_{n+1}(\mathcal{O}_\ell)$$
$$K^1 = \{(g_n, g_{n+1}) \in K \mid \det g_n \equiv 1 \pmod{\varpi}\}$$

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Easy fact:

$$\mathrm{Sh}_{\mathbf{G}}(K^1) = \mathrm{Sh}_{\mathbf{G}}(K) \times_E E[\ell]$$
$$\mathrm{Tr}_K^{K^1} \leftrightsquigarrow \mathrm{Tr}_E^{E[\ell]}$$

Tame norm relation

We are reduced to a purely local question.

Goal

For almost all split ℓ, construct

$$\phi^1 \in C_c^\infty(\mathbf{X}(F_\ell), \mathbb{Z}_p)^{K^1}$$

such that

$$\mathrm{Tr}_K^{K^1} \phi^1 = \mathscr{L} \cdot \mathbf{1}[\mathbf{X}(\mathcal{O})]$$

By applying $\Theta^{p\infty}$, this implies the main theorem.

Generalized Cartan decomposition

F now local field, ℓ now size of residue field.

Theorem (Gaitsgory–Nadler, Sakellaridis)

Let Λ^+ be the positive coweights of $\mathbf{G}$. Concretely,

$$\check{\lambda} \in \Lambda^+ \leftrightarrow (a_1 \geq \cdots \geq a_n), (b_1 \geq \cdots \geq b_{n+1}) \in \mathbb{Z}^n \times \mathbb{Z}^{n+1}$$

Then there is a decomposition

$$\mathbf{X}(F) = \bigsqcup_{\check{\lambda} \in \Lambda^+} x_{\check{\lambda}} \mathbf{G}(\mathcal{O})$$

for some explicit $x_{\check{\lambda}}$.

Image of trace

Proposition

The image of $C_c^\infty(X(F), \mathbb{Z}_p)^{K^1}$ under $\mathrm{Tr}_K^{K^1}$ is given by the divisibility conditions

$$\phi(x_{\check{\lambda}}) \in \begin{cases} \mathbb{Z}_p & \text{if all } a_i \text{ and all } b_j \text{ are distinct} \\ (\ell - 1)\mathbb{Z}_p & \text{otherwise} \end{cases}$$

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Abstract statement

The image is the set of K-invariant functions ϕ such that

$$\ell - 1 | \phi(x_{\check{\lambda}})$$

whenever $\check{\lambda}$ lies on a wall (of type T).

Unramified question

New question: Does $\mathscr{L} \cdot \mathbf{1}[\mathbf{X}(\mathcal{O})]$ satisfy the divisibility conditions?

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New input

Compute $\mathscr{L} \cdot \mathbf{1}[\mathbf{X}(\mathcal{O})]$ using the inverse *relative* Satake transform.

Relative Satake transform

Theorem (Sakellaridis)

There is an isomorphism

$$\begin{array}{ccc} C_c^\infty(\mathbf{X}(F), \mathbb{C})^K & \xrightarrow{\sim} & \mathbb{C}[A^*]^W \\ \circlearrowleft & & \circlearrowleft \\ \mathcal{H}(\mathbf{G}, \mathbb{C}) & \xrightarrow{\sim} & \mathbb{C}[A^*]^W \end{array}$$

where the bottom arrow is the usual Satake isomorphism, and the right action is multiplication.

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where the bottom arrow is the usual Satake isomorphism, and the right action is multiplication.

Let $\phi = \mathscr{L} \cdot \mathbf{1}[\mathbf{X}(\mathcal{O})]$, then

$$\hat{\phi} = \hat{\mathscr{L}} = \prod_{i,j} \left(1 - \ell^{-\frac{1}{2}} Z_i W_j\right)$$

Inverse relative Satake

Theorem (Sakellaridis)

Define the function

$$\tilde{\phi}(-) = \hat{\phi}(-^{-1}) \cdot \frac{\prod_{i_1<i_2} \left(1 - \frac{Z_{i_1}}{Z_{i_2}}\right) \prod_{j_1<j_2} \left(1 - \frac{W_{i_1}}{W_{i_2}}\right)}{\prod_{i,j} \left(1 - \ell^{-\frac{1}{2}} (Z_i W_j)^{\varepsilon_{ij}}\right)}$$

where

$$\varepsilon_{ij} = \begin{cases} +1 & \text{if } i+j \leq n+1 \\ -1 & \text{if } i+j > n+1 \end{cases}$$

Then for $\check{\lambda} \leftrightarrow (\underline{a}, \underline{b}) \in \Lambda^+$, the value $\phi(x_{\check{\lambda}})$ is the coefficient of $Z^{\underline{a}} W^{\underline{b}}$ in the power series expansion of

$$\tilde{\phi}(\ell^{-\frac{n+1-2i}{2}} Z_i, \ell^{-\frac{n+2-2j}{2}} W_j)$$

Apply the theorem to

$$\hat{\phi} = \prod_{i,j} \left(1 - \ell^{-\frac{1}{2}} Z_i W_j\right)$$

The following is immediate.

Proposition

$$\phi(x_{\check{\lambda}}) \in \mathbb{Z}[\ell^{\pm 1}]$$

This implies integrality ($\ell \neq p$), but what about divisibility by $\ell - 1$ on walls?

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Idea

"Specialize" at $\ell = 1$, i.e. show that

$$\phi(x_{\check{\lambda}})|_{\ell=1} = 0$$

if $\check{\lambda}$ is on a wall.

Simplify when $\ell = 1$

The Euler system relation specifices the Hecke operator

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This is *almost* the denominator of the kernel.

$$\begin{aligned}
\tilde{\phi}|_{\ell=1} &= \hat{\phi}(-^{-1}) \cdot \frac{\prod_{i_1<i_2}\left(1 - \frac{Z_{i_1}}{Z_{i_2}}\right)\prod_{j_1<j_2}\left(1 - \frac{W_{i_1}}{W_{i_2}}\right)}{\prod_{i,j}\left(1 - (Z_i W_j)^{\varepsilon_{ij}}\right)} \\
&= \prod_{i_1<i_2}\left(1 - \frac{Z_{i_1}}{Z_{i_2}}\right)\prod_{j_1<j_2}\left(1 - \frac{W_{i_1}}{W_{i_2}}\right) \cdot \prod_{i+j\le n+1} \frac{1 - (Z_i W_j)^{-1}}{1 - Z_i W_j} \\
&= \prod_{i_1<i_2}\left(1 - \frac{Z_{i_1}}{Z_{i_2}}\right)\prod_{j_1<j_2}\left(1 - \frac{W_{i_1}}{W_{i_2}}\right) \cdot \prod_{i+j\le n+1} \left(-(Z_i W_j)^{-1}\right)
\end{aligned}$$

The result is a polynomial in $\mathbb{C}[\underline{Z}^{\pm 1}, \underline{W}^{\pm 1}]$.

Exceptional divisibility

What happens when we exchange $Z_i \leftrightarrow Z_{i+1}$?

- First term multiplied by $-\frac{Z_i}{Z_{i+1}}$.
- Second term multiplied by $+\frac{Z_i}{Z_{i+1}}$.

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If $a_i = a_{i+1}$ in $\check{\lambda}$, then this implies automatically

$$\phi(x_{\check{\lambda}})|_{\ell=1} = 0$$

Similarly for the operation $W_j \leftrightarrow W_{j+1}$.

Speculation

Let $\mathbf{X}$ be a spherical variety for any reductive group $\mathbf{G}$.
There should be a "motivic theta element"

$$\Theta \in \mathrm{Hom}_{\mathbf{G}}(\mathrm{Fun}(\mathbf{X}(\mathbb{A}), \mathbb{Z}), \text{"integral motivic classes"})$$

- Examples/realizations should include diagonal cycles, Eisenstein classes, arithmetic theta lifts,...
- Hamiltonian induction should correspond to pushforward constructions.
- Archimedean place in our setting corresponds to choice of fixed vector in local systems.
- Arithmetic analogue of theta elements of relative Langlands program.

Euler system applications

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This uniformly recovers most known examples of Euler systems.